Problem 11170
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Proposed by E. Deutsch (USA).

Let e_n and o_n be the number of dissections of a convex n-gon by nonintersecting diagonals into an even or odd number of regions, respectively. Show that $e_n - o_n = (-1)^n$.

Solution proposed by Roberto Tauraso, Dipartimento di Matematica, Università di Roma "Tor Vergata", via della Ricerca Scientifica, 00133 Roma, Italy.

We first establish that the coefficient of $w^m z^{n-2}$ of the function

$$F(z,w) = \frac{1+z-\sqrt{1-(4w+2)z+z^2}}{2(w+1)z}$$

is the number of dissections of a convex n-gon in m polygons by nonintersecting diagonals.
Expanding $F(z,w)$ with respect to z we get

$$1 + wz + (w+2w^2)z^2 + (w+5w^2+5w^3)z^3 + (w+9w^2+21w^3+14w^4)z^4 + o(z^4)$$

and therefore $e_3 = 0$, $o_3 = 1$, $e_4 = 2$, $o_4 = 1$, $e_5 = 5$, $o_5 = 1+5 = 6$, $e_6 = 9+14 = 23$, $o_6 = 1+21 = 22$. Note that e_n and o_n are respectively the sequences A100299 and A100300 of the Encyclopedia of Integer Sequences. In order to prove the formula for $F(z,w)$ we follow the solution to exercise 7.50 in *Concrete Mathematics* by Graham-Knuth-Patashnik: each polygon has a *base* (the line at the bottom) and if $P_1, P_2, \ldots, P_{n-1}$ are $n-1$ dissected polygons, we can construct a new dissected polygon pasting the bases of each P_i to a side of n-gon different from its base. Every dissection arises in this way and by using this symbolic multiplication we get an identity between dissected polygons. If in this identity we replace each n-gon dissected in m parts with $w^m z^{n-2}$ we obtain

$$F(z,w) = 1 + \sum_{n=3}^{\infty} (wz^{n-2}) F(z,w)^{n-1} = 1 + \frac{wzF(z,w)^2}{1-zF(z,w)}$$

that is

$$(1+w)zF(z,w)^2 - (1+z)F(z,w) + 1.$$

Solving this quadratic equation and taking the appropriate solution we find the above formula.
Now we can obtain the generating function for the sequence $e_n - o_n$ simply by computing $F(z,-1)$.
We first expand $F(z,w)$ with respect to $(w-(-1)) = w+1$

$$F(z,w) = \frac{1+z-(1+z)\sqrt{1-\frac{4(w+1)z}{1+z}}}{2(w+1)z} = \frac{1+z-(1+z)\left(1-\frac{2(w+1)z}{1+z}+o(w+1)\right)}{2(w+1)z} = \frac{1}{1+z} + o(1)$$

then

$$F(z,-1) = \frac{1}{1+z} = \sum_{n=2}^{\infty} (-1)^n z^{n-2}$$

and finally $e_n - o_n = (-1)^n$. $\square$