Problem 11165

Proposed by Y. More (USA).

Let $C_k$ be the $k$th Catalan number, $\frac{1}{k+1} \binom{2k}{k}$. Prove that, for each positive integer $n$, $\sum_{k=1}^{n} C_k \equiv 1 \pmod{3}$ if and only if the base $3$ expansion of $n+1$ contains the digit $2$. Find similar characterization for the other two cases, in which the sum is congruent to $0$ or $2$ modulo $3$.

Solution proposed by Roberto Tauraso, Dipartimento di Matematica, Università di Roma “Tor Vergata”, via della Ricerca Scientifica, 00133 Roma, Italy.

First we note that if $k+1 \not\equiv 0 \pmod{3}$ then

$$C_k - C_{k-1} = \frac{1}{k+1} \binom{2k}{k} = \frac{k}{k+1} \cdot \frac{(2k)!}{k!k!} \cdot \frac{(k-1)!(k-1)!}{(2k-2)!} = \frac{4k-2}{k+1} \equiv 1 \pmod{3}$$

and

$$C_{k+1} \equiv C_k \equiv C_{k+1} \pmod{3} \text{ if } k \equiv 0 \pmod{3}.$$

Therefore, since $C_1 = 1$, then

$$\sum_{k=1}^{n} C_k \equiv \begin{cases} 1 - C_{n+1} \equiv 1 - C_n & \text{if } n \equiv 0 \pmod{3} \\ 1 & \text{if } n \equiv 1 \pmod{3} \\ 1 + C_n \equiv 1 + C_{n+1} & \text{if } n \equiv 2 \pmod{3} \end{cases}.$$

If $n \equiv 0 \pmod{3}$ then by Lucas theorem

$$C_n = \frac{1}{n+1} \binom{2n}{n} \equiv \binom{2n}{n} \equiv \binom{(2n)_i}{n_i} \cdots \binom{(2n)_1}{n_1} \binom{(2n)_0}{n_0} \pmod{3}$$

where $[(2n)_i \ldots (2n)_1(2n)_0]$ and $[n_1 \ldots n_1 n_0]$ are the ternary expansions of the numbers $2n$ and $n$. If the expansion of $n$ contains a digit $2$ then there is a carry when adding $n$ and therefore one of the binomial coefficients $\binom{(2n)_i}{n_i}$ is equal to $\binom{1}{1} = 0$ and $C_n \equiv 0 \pmod{3}$.

On the other hand if the digits of $n$ are $0$ or $1$ then $\binom{(2n)_i}{n_i} = 2(n)_i$ and $C_n \equiv \binom{2}{1} \delta(n) \equiv (-1)^{\delta(n)} \pmod{3}$ where $\delta(n)$ is the number of $1$’s in the expansion of $n$. Hence

$$\sum_{k=1}^{n} C_k \equiv \begin{cases} 1 & \text{if } n \equiv 0 \pmod{3} \text{ and the 3-expansion of } n \text{ contains the digit } 2 \\ 1 - (-1)^{\delta(n)} & \text{if } n \equiv 0 \pmod{3} \text{ and the 3-expansion of } n \text{ does not contain 2} \\ 1 & \text{if } n \equiv 1 \pmod{3} \\ 1 + (-1)^{\delta(n+1)} & \text{if } n \equiv 2 \pmod{3} \text{ and the 3-expansion of } n+1 \text{ contains the digit } 2 \\ 1 & \text{if } n \equiv 2 \pmod{3} \text{ and the 3-expansion of } n+1 \text{ does not contain 2} \end{cases}.$$

Finally we can summarize our result as follows

$$\sum_{k=1}^{n} C_k \equiv \begin{cases} 1 & \text{iff the 3-expansion of } n+1 \text{ contains the digit } 2 \\ 2 & \text{iff the 3-expansion of } n+1 \text{ does not contain 2 and the number of } 1 \text{’s is even} \\ 0 & \text{iff the 3-expansion of } n+1 \text{ does not contain 2 and the number of } 1 \text{’s is odd} \end{cases}. \square$$