Problem 11165
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Proposed by Y. More (USA).

Let C_k be the kth Catalan number, $\frac{1}{k+1}\binom{2k}{k}$. Prove that, for each positive integer n, $\sum_1^n C_k \equiv 1$ (mod 3) if and only if the base 3 expansion of $n+1$ contains the digit 2. Find similar characterization for the other two cases, in which the sum is congruent to 0 or 2 modulo 3.

Solution proposed by Roberto Tauraso, Dipartimento di Matematica, Università di Roma "Tor Vergata", via della Ricerca Scientifica, 00133 Roma, Italy.

First we note that if $k+1 \not\equiv 0 \pmod 3$ then

$$\frac{C_k}{C_{k-1}} = \frac{\frac{1}{k+1}\binom{2k}{k}}{\frac{1}{k}\binom{2(k-1)}{k-1}} = \frac{k}{k+1} \cdot \frac{(2k)!}{k!k!} \cdot \frac{(k-1)!(k-1)!}{(2k-2)!} = \frac{4k-2}{k+1} \equiv 1 \pmod 3$$

and

$$C_{k-1} \equiv C_k \equiv C_{k+1} \pmod 3 \quad \text{if } k \equiv 0 \pmod 3.$$

Therefore, since $C_1 = 1$, then

$$\sum_1^n C_k \equiv \begin{cases} 1 - C_{n+1} \equiv 1 - C_n & \text{if } n = 0 \pmod 3 \\ 1 & \text{if } n = 1 \pmod 3 \\ 1 + C_n \equiv 1 + C_{n+1} & \text{if } n = 2 \pmod 3 \end{cases}.$$

If $n = 0 \pmod 3$ then by Lucas theorem

$$C_n = \frac{1}{n+1}\binom{2n}{n} \equiv \binom{2n}{n} \equiv \binom{(2n)_t}{n_t} \cdots \binom{(2n)_1}{n_1}\binom{(2n)_0}{n_0} \pmod 3$$

where $[(2n)_t \dots (2n)_1(2n)_0]$ and $[n_t \dots n_1 n_0]$ are the ternary expansions of the numbers $2n$ and n. If the expansion of n contains a digit 2 then there is a carry when adding n and therefore one of the binomial coefficients $\binom{(2n)_i}{(n)_i}$ is equal to $\binom{1}{2} = 0$ and $C_n \equiv 0 \pmod 3$.

On the other hand if the digits of n are 0 or 1 then $(2n)_i = 2(n)_i$ and $C_n \equiv \binom{2}{1}^{\delta(n)} \equiv (-1)^{\delta(n)} \pmod 3$ where $\delta(n)$ is the number of 1's in the expansion of n. Hence

$$\sum_1^n C_k \equiv \begin{cases} 1 & \text{if } n = 0 \pmod 3 \text{ and the 3-expansion of } n \text{ contains the digit 2} \\ 1 - (-1)^{\delta(n)} & \text{if } n = 0 \pmod 3 \text{ and the 3-expansion of } n \text{ does not contain 2} \\ 1 & \text{if } n = 1 \pmod 3 \\ 1 & \text{if } n = 2 \pmod 3 \text{ and the 3-expansion of } n+1 \text{ contains the digit 2} \\ 1 + (-1)^{\delta(n+1)} & \text{if } n = 2 \pmod 3 \text{ and the 3-expansion of } n+1 \text{ does not contain 2} \end{cases}.$$

Finally we can summerize our result as follows

$$\sum_1^n C_k \equiv \begin{cases} 1 & \text{iff the 3-expansion of } n+1 \text{ contains the digit 2} \\ 2 & \text{iff the 3-expansion of } n+1 \text{ does not contain 2 and the number of 1's is even} \\ 0 & \text{iff the 3-expansion of } n+1 \text{ does not contain 2 and the number of 1's is odd} \end{cases}.$$

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