Problem 11164
(American Mathematical Monthly, Vol.112, June-July 2005)

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Show that if n is a positive integer then

$$\sum_{k=1}^{n}(-1)^{k+1}\binom{n}{k}\sum_{1\le i\le j\le k}\frac{1}{ij}=\frac{1}{n^2}.$$

Solution proposed by Roberto Tauraso, Dipartimento di Matematica, Università di Roma "Tor Vergata", via della Ricerca Scientifica, 00133 Roma, Italy.

We define for $k,r\ge 1$

$$a_{k,0}:=1,\quad a_{0,r}:=0,\quad a_{k,r}:=\sum_{1\le i_1\le i_2\le\cdots\le i_r\le k}\frac{1}{i_1 i_2\cdots i_r}$$

(note that $a_{k,1}=H_k=1+1/2+\cdots+1/k$ is the kth harmonic number).
We will prove the more general identity

$$\sum_{k=1}^{n}(-1)^{k+1}\binom{n}{k}a_{k,r}=\frac{1}{n^r}.$$

For $k,r\ge 1$

$$a_{k,r}=\sum_{i_r=1}^{k}\frac{1}{i_r}\sum_{1\le i_1\le i_2\le\cdots\le i_{r-1}\le i_r}\frac{1}{i_1 i_2\cdots i_{r-1}}=\sum_{i=1}^{k}\frac{a_{i,r-1}}{i}=a_{k-1,r}+\frac{a_{k,r-1}}{k}.$$

Therefore for $r\ge 1$ we have that

$$\begin{aligned}
\sum_{k=1}^{n}(-1)^{k+1}\binom{n}{k}a_{k,r} &= \sum_{k=1}^{n}(-1)^{k+1}\left[\binom{n-1}{k-1}+\binom{n-1}{k}\right]a_{k,r}\\
&= \sum_{k=1}^{n}(-1)^{k+1}\binom{n-1}{k-1}\left[a_{k-1,r}+\frac{a_{k,r-1}}{k}\right]+\sum_{k=1}^{n-1}(-1)^{k+1}\binom{n-1}{k}a_{k,r}\\
&= \sum_{k=1}^{n}(-1)^{k+1}\binom{n-1}{k-1}a_{k-1,r}+\frac{1}{n}\sum_{k=1}^{n}(-1)^{k+1}\binom{n}{k}a_{k,r-1}+\sum_{k=1}^{n-1}(-1)^{k+1}\binom{n-1}{k}a_{k,r}\\
&= -\sum_{k=0}^{n-1}(-1)^{k+1}\binom{n-1}{k}a_{k,r}+\frac{1}{n}\sum_{k=1}^{n}(-1)^{k+1}\binom{n}{k}a_{k,r-1}+\sum_{k=1}^{n-1}(-1)^{k+1}\binom{n-1}{k}a_{k,r}\\
&= a_{0,r}+\frac{1}{n}\sum_{k=1}^{n}(-1)^{k+1}\binom{n}{k}a_{k,r-1}=\frac{1}{n}\sum_{k=1}^{n}(-1)^{k+1}\binom{n}{k}a_{k,r-1}\,.
\end{aligned}$$

Using the above identity r times we find that

$$\sum_{k=1}^{n}(-1)^{k+1}\binom{n}{k}a_{k,r}=\frac{1}{n^r}\sum_{k=1}^{n}(-1)^{k+1}\binom{n}{k}a_{k,0}=\frac{1}{n^r}\sum_{k=1}^{n}(-1)^{k+1}\binom{n}{k}=\frac{1}{n^r}.$$

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