Problem 11161
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Proposed by E. Deutsch (USA).

Show that for all integers $n \geq 3$ the number of compositions of n into relatively prime parts is a multiple of 3. (A composition of n into k parts is a list of k positive integers that sum to n.)

Solution proposed by Roberto Tauraso, Dipartimento di Matematica, Università di Roma "Tor Vergata", via della Ricerca Scientifica, 00133 Roma, Italy.

The number of compositions of n into k parts is the number of positive integral solutions of the equation $x_1+x_2+\cdots+x_k=n$ which is equal to $\binom{n-1}{k-1}$. Therefore the total number of compositions of n is

$$t(n)=\sum_{k=2}^{n}\binom{n-1}{k-1}=2^{n-1}-1.$$

Let $c_d(n)$ be the number of compositions of n such that the greatest common divisor of its parts is equal to d. We want to prove that $c_1(n)\equiv 0 \pmod 3$ for $n\geq 3$. Since

$$x_1+\cdots+x_k=n \text{ and } \gcd(x_1,\ldots,x_k)=d \quad \text{iff} \quad \tfrac{x_1}{d}+\cdots+\tfrac{x_k}{d}=\tfrac{n}{d} \text{ and } \gcd(\tfrac{x_1}{d},\ldots,\tfrac{x_k}{d})=1$$

then $c_d(n)=c_1(n/d)$ for any divisor d of n. Hence

$$t(n)=\sum_{d|n}c_d(n)=\sum_{d|n}c_1(n/d)=\sum_{d|n}c_1(d)$$

and by the Möbius Inversion Formula

$$c_1(n)=\sum_{d|n}\mu(n/d)t(d)=\sum_{d|n}\mu(n/d)(2^{d-1}-1)\equiv\sum_{d|n}\mu(n/d)((-1)^{d-1}-1)\equiv\sum_{d|n}\mu(n/d)p(d) \pmod 3$$

where $p(n)$ is the characteristic function of the set of the even numbers

$$p(n)=\begin{cases}1 & \text{if } n \text{ is even}\\ 0 & \text{if } n \text{ is odd}\end{cases}.$$

Since $p(n)=\sum_{d|n}\delta_{d,2}$, where $\delta_{d,2}$ is the characteristic function of the point 2, then by the Möbius Inversion Formula

$$\delta_{n,2}=\sum_{d|n}\mu(n/d)p(d)$$

and finally $c_1(n)\equiv\delta_{n,2} \pmod 3$. □