**Problem 11161**  

Proposed by E. Deutsch (USA).

Show that for all integers \( n \geq 3 \) the number of compositions of \( n \) into relatively prime parts is a multiple of 3. (A composition of \( n \) into \( k \) parts is a list of \( k \) positive integers that sum to \( n \).)

Solution proposed by Roberto Tauraso, Dipartimento di Matematica, Università di Roma “Tor Vergata”, via della Ricerca Scientifica, 00133 Roma, Italy.

The number of compositions of \( n \) into \( k \) parts is the number of positive integral solutions of the equation \( x_1 + x_2 + \cdots + x_k = n \) which is equal to \( \binom{n-1}{k-1} \). Therefore the total number of compositions of \( n \) is

\[
t(n) = \sum_{k=2}^{n} \binom{n-1}{k-1} = 2^{n-1} - 1.
\]

Let \( c_d(n) \) be the number of compositions of \( n \) such that the greatest common divisor of its parts is equal to \( d \). We want to prove that \( c_1(n) \equiv 0 \pmod{3} \) for \( n \geq 3 \). Since

\[
x_1 + \cdots + x_k = n \quad \text{and} \quad \gcd(x_1, \ldots, x_k) = d \iff \frac{x_1}{d} + \cdots + \frac{x_k}{d} = \frac{n}{d} \quad \text{and} \quad \gcd(\frac{x_1}{d}, \ldots, \frac{x_k}{d}) = 1
\]

then \( c_d(n) = c_1(n/d) \) for any divisor \( d \) of \( n \). Hence

\[
t(n) = \sum_{d|n} c_d(n) = \sum_{d|n} c_1(n/d) = \sum_{d|n} c_1(d)
\]

and by the Möbius Inversion Formula

\[
c_1(n) = \sum_{d|n} \mu(n/d)t(d) = \sum_{d|n} \mu(n/d)(2^{d-1} - 1) \equiv \sum_{d|n} \mu(n/d)((-1)^{d-1} - 1) \equiv \sum_{d|n} \mu(n/d)p(d) \pmod{3}
\]

where \( p(n) \) is the characteristic function of the set of the even numbers

\[
p(n) = \begin{cases} 
1 & \text{if } n \text{ is even} \\
0 & \text{if } n \text{ is odd}
\end{cases}
\]

Since \( p(n) = \sum_{d|n} \delta_{d,2} \), where \( \delta_{d,2} \) is the characteristic function of the point 2, then by the Möbius Inversion Formula

\[
\delta_{n,2} = \sum_{d|n} \mu(n/d)p(d)
\]

and finally \( c_1(n) \equiv \delta_{n,2} \pmod{3} \). \( \square \)