Problem 11158
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Proposed by D. Beckwith (USA).

Let n be a positive integer, and let p be a prime number. Prove that $p^p \mid n!$ implies that $p^{p+1} \mid n!$.

Solution proposed by Radouan Boukharfane, Polytechnique de Montreal, Canada, and Roberto Tauraso, Dipartimento di Matematica, Università di Roma "Tor Vergata", Italy.

It is well known that that the power of the prime p in the factorization of $n!$ is given by the sum

$$\alpha_p(n) = \sum_{k=1}^{\lfloor \log_p(n) \rfloor} \left\lfloor \frac{n}{p^k} \right\rfloor$$

(see for example *Problem Solving Through Problems* by L.C. Larson at p.104).
We consider two cases:
(1) if $1 \leq n < p^2$ then $\log_p(n) < 2$ and

$$\alpha_p(n) \leq \left\lfloor \frac{n}{p} \right\rfloor < p$$

(2) if $n \geq p^2$ then $\log_p(n) \geq 2$ and

$$\alpha_p(n) \geq \left\lfloor \frac{n}{p} \right\rfloor + \left\lfloor \frac{n}{p^2} \right\rfloor \geq p+1$$

Therefore if $p^p \mid n!$ then $\alpha_p(n) \geq p$ and by the above discussion $\alpha_p(n) \geq p+1$, that is $p^{p+1} \mid n!$. $\square$