Problem 11153
(American Mathematical Monthly, Vol.112, May 2005)

Proposed by P. Bracken (USA).

Let $x_1 = 1$, and for $n \geq 1$ let $x_{n+1} = x_n + 2 + 1/x_n$. If $y_n = 2n + (1/2)\log(n) - x_n$, show that the sequence $\langle y_n \rangle$ is eventually increasing.

Solution proposed by Giulio Francot and Roberto Tauraso,
Dipartimento di Matematica, Università di Roma "Tor Vergata", via della Ricerca Scientifica, 00133 Roma, Italy.

Let $n \geq 3$ then summing the equations $x_{k+1} = x_k + 2 + 1/x_k$ for $k = 1, \ldots, n-1$ we obtain

$$x_n = x_1 + 2(n-1) + \sum_{k=1}^{n-1} \frac{1}{x_k} = 2n + \sum_{k=2}^{n-1} \frac{1}{x_k}.$$

Therefore the sequence $\langle x_n \rangle$ is positive and increasing. Moreover for $n \geq 3$

$$2n < x_n = x_{n-1} + 2 + 1/x_{n-1} \leq x_{n-1} + 3 \leq 3(n-1) + 1 < 3n.$$

The following is a better lower estimate

$$x_n = 2n + \sum_{k=2}^{n-1} \frac{1}{x_k} > 2n + \sum_{k=2}^{n-1} \frac{1}{3k} = 2n + \frac{1}{3}\left(H_{n-1} - 1\right).$$

Now we consider the sequence $\langle y_n \rangle$. Since

$$y_{n+1} - y_n = 2 + \frac{1}{2}\log\left(\frac{n+1}{n}\right) - (x_{n+1} - x_n) = \frac{1}{2}\log\left(1 + \frac{1}{n}\right) - \frac{1}{x_n}$$

then $\langle y_n \rangle$ is eventually increasing if the following inequality eventually holds

$$x_n > \frac{2}{\log\left(1 + \frac{1}{n}\right)}.$$

We note that

$$\begin{aligned}\frac{2}{\log\left(1+\frac{1}{n}\right)} &= \frac{2}{\frac{1}{n} - \frac{1}{2n^2} + o\left(\frac{1}{n^2}\right)} = \frac{2n}{1 - \frac{1}{2n} + o\left(\frac{1}{n}\right)} \\ &= 2n\left(1 + \frac{1}{2n} + o\left(\frac{1}{n}\right)\right) = 2n + 1 + o(1).\end{aligned}$$

Finally by the lower estimate, since H_n goes to infinity, eventually

$$x_n > 2n + \frac{1}{3}\left(H_{n-1} - 1\right) \geq 2n + 1 + o(1).$$

□