Problem 11152
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Proposed by M. Ivan (Romania).

Evaluate

$$I=\int_0^1 \frac{\log(\cos(\pi x/2))}{x(1+x)}\,dx.$$

Solution proposed by Roberto Tauraso, Dipartimento di Matematica, Università di Roma "Tor Vergata", via della Ricerca Scientifica, 00133 Roma, Italy.

First we note that the integral can be decomposed in this way

$$I=\int_0^1 \frac{\log(\cos(\pi x/2))}{x}\,dx+\int_0^1 \frac{\log(\cos(\pi x/2))}{1+x}\,dx=\int_0^1 \frac{\log(\cos(\pi x/2))}{x}\,dx-\int_1^2 \frac{\log(\sin(\pi x/2))}{x}\,dx.$$

Now we show that a more general identity holds: for any integer $n\geq 0$

$$I=-n(\log 2)^2+\int_0^{1/2^n} \frac{\log(\cos(\pi x/2))}{x}\,dx-\int_{1/2^n}^{1/2^{n-1}} \frac{\log(\sin(\pi x/2))}{x}\,dx.$$

We already proved the identity for $n=0$. The inductive step goes as follows:

$$\begin{aligned}\log(\sin(\pi x/2)) &= \log(\sin(2\pi x/4))=\log(2\sin(\pi x/4)\cos(\pi x/4))\\ &= \log 2+\log(\sin(\pi x/4))+\log(\cos(\pi x/4))\end{aligned}$$

and therefore

$$\begin{aligned}\int_{1/2^n}^{1/2^{n-1}} \frac{\log(\sin(\pi x/2))}{x}\,dx &= \log 2\int_{1/2^n}^{1/2^{n-1}} \frac{1}{x}\,dx+\int_{1/2^n}^{1/2^{n-1}} \frac{\log(\sin(\pi x/4))}{x}\,dx+\int_{1/2^n}^{1/2^{n-1}} \frac{\log(\cos(\pi x/4))}{x}\,dx\\ &= (\log 2)^2+\int_{1/2^{n+1}}^{1/2^n} \frac{\log(\sin(\pi x/2))}{x}\,dx+\int_{1/2^{n+1}}^{1/2^n} \frac{\log(\cos(\pi x/2))}{x}\,dx.\end{aligned}$$

Hence

$$I=-(n+1)(\log 2)^2+\int_0^{1/2^{n+1}} \frac{\log(\cos(\pi x/2))}{x}\,dx-\int_{1/2^{n+1}}^{1/2^n} \frac{\log(\sin(\pi x/2))}{x}\,dx.$$

Moreover, since

$$\sin(\pi x/2)=\pi x/2+x^3 f(x)=(\pi x/2)(1+2x^2 f(x)/\pi)$$

where $f(x)$ is a continuous bounded map in the interval $(0,2)$, then

$$\begin{aligned}\int_{1/2^n}^{1/2^{n-1}} \frac{\log(\sin(\pi x/2))}{x}\,dx &= \int_{1/2^n}^{1/2^{n-1}} \frac{\log(\pi x/2)}{x}\,dx+\int_{1/2^n}^{1/2^{n-1}} \frac{\log(1+2x^2 f(x)/\pi)}{x}\,dx\\ &= \frac{1}{2}\left[(\log(\pi x/2))^2\right]_{1/2^n}^{1/2^{n-1}}+\int_{1/2^n}^{1/2^{n-1}} \frac{\log(1+2x^2 f(x)/\pi)}{x}\,dx\\ &= \log(2)\log(\pi/\sqrt{2})-n(\log 2)^2+\int_{1/2^n}^{1/2^{n-1}} \frac{\log(1+2x^2 f(x)/\pi)}{x}\,dx\end{aligned}$$

Finally

$$I=-\log(2)\log(\pi/\sqrt{2})+\int_0^{1/2^n} \frac{\log(\cos(\pi x/2))}{x}\,dx-\int_{1/2^n}^{1/2^{n-1}} \frac{\log(1+2x^2 f(x)/\pi)}{x}\,dx$$

and, taking the limit as n goes to infinity, we find that the two integrals vanish because the integrand functions are bounded in the corresponding shrinking intervals. Hence

$$I = -\log(2)\log(\pi/\sqrt{2}) \approx -0.5532397859.$$

$\square$