Problem 11151
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Proposed by D. Knuth (USA).

Suppose n people are sitting at a circular table. Let $e_{n,k}$ denote the number of ways to partition them in k affinity groups with no two members of a group seated next to each other. For example $e_{4,3} = 2$, $e_{5,3} = 5$, and $e_{6,3} = 10$. For $k \geq 2$ find the generating function $f_k(z) = \sum_{n=0}^{\infty} e_{n,k} z^n$.

Solution proposed by Roberto Tauraso, Dipartimento di Matematica, Università di Roma "Tor Vergata", via della Ricerca Scientifica, 00133 Roma, Italy.

We easily note that

$$f_2(z) = z^2 + z^4 + z^6 + z^8 + \cdots = \sum_{j=1}^{\infty} z^{2j} = \frac{z^2}{1-z^2}.$$

We consider the number of ways to partition n persons in k affinity groups at a *linear* table. It is the sum of the number of partitions with the 1st and the nth persons in different groups, that is just $e_{n,k}$, and the number of partitions with the 1st and the nth persons in the same group, say $d_{n,k}$. Assume that $n \geq k \geq 3$, then in order to compute $e_{n,k}$ we either put the nth person by itself or we distribute him in one of the affinity groups of $n-1$ persons which does not contain the 1st person or the $(n-1)$th person:

$$e_{n,k} = [e_{n-1,k-1} + d_{n-1,k-1}] + [(k-2)\cdot e_{n-1,k} + (k-1)\cdot d_{n-1,k}]\,.$$

In order to compute $d_{n,k}$ we put the nth person in the same group of the 1st one provided it does not contain the $(n-1)$th person:

$$d_{n,k} = 1 \cdot e_{n-1,k} + 0 \cdot d_{n-1,k} = e_{n-1,k}.$$

Hence, eliminating $d_{n,k}$, we find the following recurrence for $n \geq k \geq 3$

$$e_{n,k} = e_{n-1,k-1} + e_{n-2,k-1} + (k-2)e_{n-1,k} + (k-1)e_{n-2,k}.$$

Since $e_{n,k} = 0$ for $k > n$, multiplying this equation by z^n and summing up for n from 3 to ∞ we get

$$f_k(z) = (z+z^2)f_{k-1}(z) + \left((k-2)z + (k-1)z^2\right) f_k(z)$$

that is, for $k \geq 3$,

$$f_k(z) = \frac{(z+z^2)f_{k-1}(z)}{1-(k-2)z-(k-1)z^2} = \frac{z(1+z)f_{k-1}(z)}{(1+z)^2 - kz(1+z)} = \frac{zf_{k-1}(z)}{1-(k-1)z}.$$

Finally

$$f_k(z) = \left(\prod_{j=3}^{k} \frac{z}{1-(j-1)z}\right) f_2(z) = \frac{z}{1+z}\prod_{j=1}^{k-1}\frac{z}{1-jz}.$$

For example

$$f_3(z) = \frac{z^3}{(1-z^2)(1-2z)} = z^3 + 2z^4 + 5z^5 + 10z^6 + 21z^7 + 42z^8 + 85z^9 + \cdots\,.$$

□

Remark. It is interesting to note that

$$f_k(z) = \frac{z}{1+z}\, g_{k-1}(z) \quad k \geq 2$$

where

$$g_k(z) = \sum_{n=0}^{\infty} \left\{ {n \atop k} \right\} z^n = \prod_{j=1}^{k} \frac{z}{1-jz}$$

is the generating function of the *Stirling numbers of the second kind*, i. e. the numbers of partitions of a set of n elements in k non-empty subsets (see *Concrete Mathematics* by Graham-Knuth-Patashnik).