Problem 11149
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Let $a > 0$. Find

$$\lim_{n\to\infty} n\log(1+\log(1+(\cdots\log(1+a/n)\cdots)))$$

where the parentheses are nested to depth n.

Solution proposed by Roberto Tauraso, Dipartimento di Matematica, Università di Roma "Tor Vergata", via della Ricerca Scientifica, 00133 Roma, Italy.

Let $f(z)$ be the principal complex logarithm $\log(1+z)$. We denote by $f_n(z)$ the nth-iterate of f in the right half plane $H = \{z \in \mathbb{C} : \mathrm{Re}(z) > 0\}$ (note that $f(H) \subset H$). We will prove that

$$\lim_{n\to\infty} nf_n(z/n) = \frac{2z}{z+2} \quad \text{for } z \in H$$

(which is the required limit when z is a real positive number).

Since $|f(z)| \le |z|$ for $z \in H$ (see the final remark) then

$$|nf_n(z/n)| = n|f(f_{n-1}(z/n))| \le n|f_{n-1}(z/n)| \le n|z/n| = |z|$$

and therefore the sequence of analytic functions $\{nf_n(z/n)\}$ is locally bounded on H. If we show that this sequence converges to the analytic function $g(z) = 2z/(z+2)$ in $(0,2)$ (which has an accumulation point in H), then by Vitali's theorem the sequence converges uniformly on compact sets of H to g.
Let $\sum_{k=1}^{\infty} a_{n,k}z^k$ be the expansion of $f_n(z)$ at zero. Then $a_{1,k} = (-1)^{k+1}/k$ and, since $f_n = f_{n-1}\circ f$ and $f^{(k)}(0) = (-1)^{k+1}(k-1)!$, by the Faà di Bruno's theorem we are able find a recursion formula for the coefficients for $n \ge 2$:

$$\begin{aligned} a_{n,k} &= \frac{f_n^{(k)}(0)}{k!} = \sum_{\pi(k)} \frac{\left[f^{(1)}(0)\right]^{r_1}\left[f^{(2)}(0)\right]^{r_2}\cdots\left[f^{(k)}(0)\right]^{r_k}}{r_1!r_2!\cdots r_k!\cdot(1!)^{r_1}(2!)^{r_2}\cdots(k!)^{r_k}} \cdot f_{n-1}^{(r)}(0) \\ &= \sum_{\pi(k)} \binom{r}{r_1, r_2, \ldots, r_k} \cdot \frac{(-1)^{k+r}}{1^{r_1}2^{r_2}\cdots k^{r_k}} \cdot a_{n-1,r} \end{aligned}$$

where the sum runs over all partitions $\pi(k)$ of the positive integer k: $k = 1r_1 + 2r_2 + \cdots + kr_k$, r_i denotes the number of parts of size i, and $r = r_1 + r_2 + \cdots + r_k$ is the total number of parts.
It is easy to determine an explicit formula for the first coefficients:

$$\begin{aligned} a_{n,1} &= a_{n-1,1} = a_{1,1} = 1 \\ a_{n,2} &= a_{n-1,2} - a_{n-1,1}/2 = a_{n-1,2} - 1/2 = -n/2 \\ a_{n,3} &= a_{n-1,3} - a_{n-1,2} + a_{n-1,1}/3 = a_{n-1,3} - (n-1)/2 + 1/3 = n^2/4 + n/12. \end{aligned}$$

Now we show by induction with respect to k that

$$a_{n,k} = \left(-\frac{n}{2}\right)^{k-1} + Q_k(n)$$

where Q_k is a polynomial of degree $k-2$. The inductive step: by the recursion formula

$$a_{n,k} = a_{n-1,k} - \binom{k-1}{2} a_{n-1,k-1} + \sum_{r=1}^{k-2} c_{k,r}\, a_{n-1,r}$$

where the first two terms are associated to the partitions $1 \cdot k$ and $1 \cdot (k-2) + 2 \cdot 1$. Hence

$$\begin{aligned} a_{n,k} &= a_{1,k} + \sum_{j=1}^{n-1} \left(-\binom{k-1}{2} a_{j,k-1} + \sum_{r=1}^{k-2} c_{k,r}\, a_{j,r} \right) \\ &= a_{1,k} + \sum_{j=1}^{n-1} \left(-\binom{k-1}{2} \left(\left(-\frac{j}{2}\right)^{k-2} + Q_{k-1}(j) \right) + \sum_{r=1}^{k-2} c_{k,r}\, a_{j,r} \right) \\ &= (k-1)\left(-\frac{1}{2}\right)^{k-1} \sum_{j=1}^{n-1} j^{k-2} + O\left(\sum_{j=1}^{n-1} j^{k-3} \right) = \left(-\frac{1}{2}\right)^{k-1} n^{k-1} + O(n^{k-2}). \end{aligned}$$

Finally

$$n f_n(z/n) = \sum_{k=1}^{\infty} \frac{a_{n,k}}{n^{k-1}} \cdot z^k = \sum_{k=1}^{\infty} \left(\left(-\frac{1}{2}\right)^{k-1} + \frac{Q_k(n)}{n^{k-1}} \right) z^k$$

and for $z \in (0,2)$ taking the limit as n goes to infinity we have that

$$\lim_{n\to\infty} n f_n(z/n) = z \sum_{k=1}^{\infty} \left(-\frac{z}{2}\right)^{k-1} = \frac{z}{1-(-z/2)} = \frac{2z}{2+z}.$$

□

Remark. $|\log(1+z)| \le |z|$ for all z in the right half-plane H.
The function $h(z) = \log(1+z)/z$ is analytic in a neighborhood of H then by the maximum modulus theorem it suffices to show that $|h(z)| \le 1$ on the boundary of the domain $H \cap \{|z| < r\}$ for r greater than some r_0. On the imaginary axes $z = iy$:

$$|\log(1+z)|^2 = |(1/2)\log(1+y^2) + i \arctan y|^2 = (1/4)(\log(1+y^2))^2 + (\arctan y)^2 \le y^2 = |z|^2.$$

Moreover, on the semicircumference $z = re^{i\theta}$ with $\theta \in [-\pi/2, \pi/2]$:

$$|\log(1+z)| \le \log|1+z| + \pi/2 \le \log(1+r) + \pi/2 \le (1/2)(r-1) + \log 2 + \pi/2 \le r = |z|$$

for $r \ge r_0 = 2\log 2 + \pi - 1$.