Problem 11145
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Proposed by J. Zinn (USA).

Find the least c such that if $n \geq 1$ and $a_1, \dots, a_n > 0$ then

$$\sum_{k=1}^{n} \frac{k}{\sum_{j=1}^{k} 1/a_j} \leq c \sum_{k=1}^{n} a_k.$$

Solution proposed by Roberto Tauraso, Dipartimento di Matematica, Università di Roma "Tor Vergata", via della Ricerca Scientifica, 00133 Roma, Italy.

Let $h_k = k / \sum_{j=1}^{k} 1/a_j$ i.e. the harmonic mean of $a_1, \dots, a_k$. If we take $a_k = 1/k$ we have that

$$\sum_{k=1}^{n} h_k = \sum_{k=1}^{n} \frac{k}{1 + \cdots + k} = \sum_{k=1}^{n} \frac{2}{k+1} = 2\left(H_n - \frac{n}{n+1}\right)$$

where $H_n = \sum_{k=1}^{n} 1/k$ is the n-th harmonic number. Hence

$$c \geq \left(\sum_{k=1}^{n} h_k\right) / \left(\sum_{k=1}^{n} a_k\right) = 2\left(1 - \frac{n}{(n+1)H_n}\right),$$

and, since H_n goes to infinity, the above inequality holds for every positive integer n only if the constant c is greater or equal to 2.
Now we prove that it actually holds for $c = 2$ and therefore this is just the best constant. Since

$$h_k^2 + h_{k-1}^2 \geq 2 h_k h_{k-1}$$

then

$$\begin{aligned} h_k - \frac{h_k^2}{2a_k} &= h_k - \frac{h_k^2}{2}\left(\frac{k}{h_k} - \frac{k-1}{h_{k-1}}\right) = h_k\left(1 - \frac{k}{2}\right) + \frac{k-1}{2}\frac{h_k^2}{h_{k-1}} \\ &\geq h_k\left(1 - \frac{k}{2}\right) + \frac{k-1}{2}\left(2h_k - h_{k-1}\right) = \frac{1}{2}\left(k h_k - (k-1) h_{k-1}\right). \end{aligned}$$

Summing up for $k = 1, \dots, n$ (let $h_0 = 0$) we find that

$$\sum_{k=1}^{n} h_k - \frac{1}{2}\sum_{k=1}^{n} \frac{h_k^2}{a_k} \geq \frac{1}{2}\sum_{k=1}^{n} \left(k h_k - (k-1) h_{k-1}\right) = \frac{1}{2} n h_n \geq 0$$

hence

$$2\sum_{k=1}^{n} h_k \geq \sum_{k=1}^{n} \left(\frac{h_k}{\sqrt{a_k}}\right)^2.$$

After multipling by $\sum_{k=1}^{n} a_k$, by Cauchy's inequality we obtain

$$2\sum_{k=1}^{n} a_k \cdot \sum_{k=1}^{n} h_k \geq \sum_{k=1}^{n} (\sqrt{a_k})^2 \cdot \sum_{k=1}^{n} \left(\frac{h_k}{\sqrt{a_k}}\right)^2 \geq \left(\sum_{k=1}^{n} h_k\right)^2$$

that is

$$2\sum_{k=1}^{n} a_k \geq \sum_{k=1}^{n} h_k.$$

It is interesting to remark that the proposed inequality is a variation (take $p = -1$) of a result due to G. H. Hardy (see for example the book *Inequalities* of Hardy-Littlewood-Polya): for $p > 1$

$$\sum_{k=1}^{n} \left(\frac{1}{k} \sum_{j=1}^{k} a_j^{1/p} \right)^p \leq \left(\frac{p}{p-1} \right)^p \sum_{k=1}^{n} a_k .$$

□