Let $[\binom{n}{m}]$ be (the absolute value of) the $(n,m)$th Stirling number of the first kind, namely, the number of permutations of $n$ objects having $m$ cycles. Given that $n$ is a positive integer and that $[\binom{n}{m}] = \max_{1 \leq k \leq n} [\binom{n}{k}] n^k$, prove that

$$n \log 2 - \frac{3}{4} < m < n \log 2 + \frac{4}{3}.$$

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We will use the following result due to J. N. Darroch (see the Computer Musing’s video "Hooray for Probability Theory" by Knuth):

Let $P(z) = \sum_{k=0}^{n} a_k z^k$ be a polynomial of degree $n \geq 1$ with non-negative coefficients such that all roots are real. Then the sequence of the coefficients is unimodal and has either a unique index $m$ or two consecutive indices $m$ such that $a_m = \max_{0 \leq k \leq n} a_k$. Such a mode $m$ differs from the mean $\mu = P'(1)/P(1)$ less than 1. More precisely

$$a_0 < a_1 < \cdots < a_{\lfloor \mu \rfloor} \quad \text{and} \quad a_{\lceil \mu \rceil} > \cdots > a_{n-1} > a_n.$$

The Stirling numbers of the first kind satisfy the following identity

$$x^n = \sum_{k=0}^{n} \binom{n}{k} x^k.$$

Therefore the polynomial

$$P(x) = (nx)(nx + 1)(nx + 2) \cdots (nx + n - 1) = (nx)^n = \sum_{k=0}^{n} \binom{n}{k} n^k x^k$$

has non-negative coefficients and all roots are real. Hence by Darroch’s theorem if $a_m = \max_{0 \leq k \leq n} a_k$ then $\mu - 1 < \lfloor \mu \rfloor \leq m \leq \lceil \mu \rceil < \mu + 1$ where

$$\mu = \frac{P'(1)}{P(1)} = n \left( \frac{1}{n} + \frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{2n-1} \right) = n \left( H_{2n} - H_n \right) + \frac{1}{2}.$$

Since for $n \geq 1$ there is $0 < \epsilon_n < 1$ such that

$$H_n = \log n + \gamma + \frac{1}{2n} + \frac{\epsilon_n}{12n^2} + \frac{\epsilon_n}{120n^4}$$

(see Concrete Mathematics by Graham-Knuth-Patashnik) then

$$\mu = n \log 2 + \frac{1}{4} + \frac{3}{48n} + \frac{\epsilon_n^2}{1920n^6} - \frac{\epsilon_n}{120n^3}.$$

Finally

$$\mu + 1 < n \log 2 + \frac{5}{4} + \frac{3}{48n} + \frac{1}{1920n^3} < n \log 2 + \frac{2521}{1920} \quad \text{and} \quad n \log 2 + \frac{4}{3}$$

and

$$\mu - 1 > n \log 2 - \frac{3}{4} + \frac{3}{48n} - \frac{1}{120n^3} > n \log 2 - \frac{3}{4}.$$