Problem 11142
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Proposed by D. Knuth (USA).

Let $\left[{n \atop m}\right]$ *be (the absolute value of) the* (n,m)*th Stirling number of the first kind, namely, the number of permutations of* n *objects having* m *cycles. Given that* n *is a positive integer and that* $\left[{n \atop m}\right]n^m = \max_{1\leq k\leq n}\left[{n \atop k}\right]n^k$*, prove that*

$$n\log 2 - \frac{3}{4} < m < n\log 2 + \frac{4}{3}.$$

Solution proposed by Roberto Tauraso, Dipartimento di Matematica, Università di Roma "Tor Vergata", via della Ricerca Scientifica, 00133 Roma, Italy.

We will use the following result due to J. N. Darroch (see the Computer Musing's video *Hooray for Probability Theory* by Knuth):

Let $P(z) = \sum_{k=0}^n a_k z^k$ *be a polynomial of degree* $n \geq 1$ *with non-negative coefficients such that all roots are real. Then the sequence of the coefficients is* unimodal *and has either a unique index* m *or two consecutive indices* m *such that* $a_m = \max_{0\leq k\leq n} a_k$*. Such a mode* m *differs from the* mean $\mu = P'(1)/P(1)$ *less than* 1*. More precisely*

$$a_0 < a_1 < \cdots < a_{\lfloor\mu\rfloor} \quad \text{and} \quad a_{\lceil\mu\rceil} > \cdots > a_{n-1} > a_n.$$

The Stirling numbers of the first kind satisfy the following identity

$$x^{\overline{n}} = \sum_{k=0}^n \left[{n \atop k}\right] x^k.$$

Therefore the polynomial

$$P(x) = (nx)(nx+1)(nx+2)\cdots(nx+n-1) = (nx)^{\overline{n}} = \sum_{k=0}^n \left[{n \atop k}\right] n^k x^k$$

has non-negative coefficients and all roots are real. Hence by Darroch's theorem if $a_m = \max_{0\leq k\leq n} a_k$ then $\mu - 1 < \lfloor\mu\rfloor \leq m \leq \lceil\mu\rceil < \mu + 1$ where

$$\mu = \frac{P'(1)}{P(1)} = n\left(\frac{1}{n} + \frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{2n-1}\right) = n\left(H_{2n} - H_n\right) + \frac{1}{2}.$$

Since for $n \geq 1$ there is $0 < \epsilon_n < 1$ such that

$$H_n = \log n + \gamma + \frac{1}{2n} - \frac{1}{12n^2} + \frac{\epsilon_n}{120n^4}$$

(see *Concrete Mathematics* by Graham-Knuth-Patashnik) then

$$\mu = n\log 2 + \frac{1}{4} + \frac{3}{48n} + \frac{\epsilon_{n^2}}{1920n^3} - \frac{\epsilon_n}{120n^3}.$$

Finally

$$\mu + 1 < n\log 2 + \frac{5}{4} + \frac{3}{48n} + \frac{1}{1920n^3} \leq n\log 2 + \frac{2521}{1920} < n\log 2 + \frac{4}{3}$$

and

$$\mu - 1 > n\log 2 - \frac{3}{4} + \frac{3}{48n} - \frac{1}{120n^3} > n\log 2 - \frac{3}{4}.$$

□