Problem 11139
(American Mathematical Monthly, Vol.112, March 2005)

Proposed by G. Bennett (USA).

Show that if $p > 1$ *or* $p < 0$, *then*

$$\frac{1^p}{3^p} < \frac{1^p + 3^p}{5^p + 7^p} < \frac{1^p + 3^p + 5^p}{7^p + 9^p + 11^p} < \cdots < \frac{1^p + \cdots + (2n-1)^p}{(2n+1)^p + \cdots + (4n-1)^p} < \cdots$$

while if $0 < p < 1$ *the inequalities are reversed.*

Solution proposed by Roberto Tauraso, Dipartimento di Matematica, Università di Roma "Tor Vergata", via della Ricerca Scientifica, 00133 Roma, Italy.

For $n \geq 1$, let

$$s_p(n) = 1^p + \cdots + (2n-1)^p = \sum_{k=0}^{n-1} (2k+1)^p$$

then we have to show that the sequence $s_p(n)/(s_p(2n) - s_p(n))$ is strictly increasing for $p \notin [0,1]$ and strictly decreasing for $p \in (0,1)$ (it is constant for $p = 0$ and $p = 1$).
Assume that $p \notin [0,1]$ (if $p \in (0,1)$ all the inequalities are reversed) then

$$\frac{s_p(n)}{s_p(2n) - s_p(n)} < \frac{s_p(n+1)}{s_p(2(n+1)) - s_p(n+1)}$$

if and only if

$$s_p(2n+2) s_p(n) < s_p(2n) s_p(n+1)$$

that is

$$\left(s_p(2n) + (4n+1)^p + (4n+3)^p\right) s_p(n) < s_p(2n) \left(s_p(n) + (2n+1)^p\right)$$

and therefore

$$\left((4n+1)^p + (4n+3)^p\right) \sum_{k=0}^{n-1} (2k+1)^p < (2n+1)^p \sum_{k=0}^{n-1} \left((4k+1)^p + (4k+3)^p\right).$$

The last inequality holds if for all $k = 0, \ldots, n-1$

$$\left((4n+1)^p + (4n+3)^p\right)(2k+1)^p < (2n+1)^p \left((4k+1)^p + (4k+3)^p\right)$$

or

$$\frac{(4n+1)^p + (4n+3)^p}{(2n+1)^p} < \frac{(4k+1)^p + (4k+3)^p}{(2k+1)^p}$$

which is true because if $p \notin [0,1]$ then the function

$$f_p(x) = \frac{(4x+1)^p + (4x+3)^p}{(2x+1)^p} = 2^p \frac{(x+1/4)^p + (x+3/4)^p}{(x+1/2)^p}$$

is strictly decreasing for $x \geq 0$ (it is strictly increasing if $p \in (0,1)$). This property of $f_p(x)$ can be easily verified considering the sign of its derivative which is positive if and only if

$$p(x+1/4)^{p-1} > p(x+3/4)^{p-1}.$$

$\square$