Problem 11138
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Proposed by R. Tauraso (Italy).

For $n \geq 2$ consider the region obtained by removing from the square $[0, 2n] \times [0, 2n]$ the four $(n-2) \times (n-2)$ squares centered at the points $(n \pm n/2, n \pm n/2)$. Find the number of domino tilings of this region, and show that it is a perfect square.

Solution proposed by Roberto Tauraso, Dipartimento di Matematica, Università di Roma "Tor Vergata", via della Ricerca Scientifica, 00133 Roma, Italy.

We will show that the total number of domino tilings is

$$T_n = (2 \cdot f_n^2 + 3 + (-1)^n)^2$$

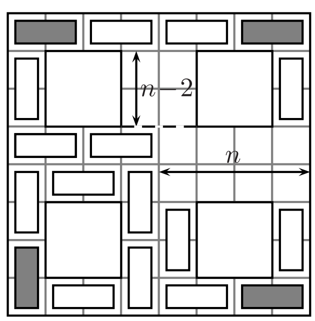
where f_n is the n-th Fibonacci number. The first terms of the sequence are:

$$T_2 = 6^2 = 36\,,\ T_3 = 10^2 = 100\,,\ T_4 = 22^2 = 484\,,\ T_5 = 52^2 = 2704.$$

There are a lot of examples of grids where the total number of domino tilings is a perfect square (see [2] for a general result). The most famous one is the $2n \times 2n$ square which gives a perfect square if and only if n is even (sequence $A004003$ in [3]).

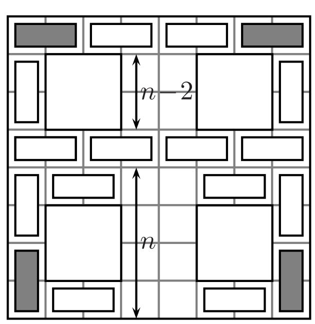
The main result that we are going to use is that the number of domino tilings of a $2 \times n$ rectangle is f_{n+1} (see [1]). We assume that n is even (for n odd the discussion is very similar) and we start by covering the corners (black dominoes). The orientation of these four dominoes forces a part of the tiling (white dominoes). A dashed segment represents a fault line. Up to rotations and reflections it suffices to consider the following four configurations.

1. There are 8 tilings equivalent to this one:



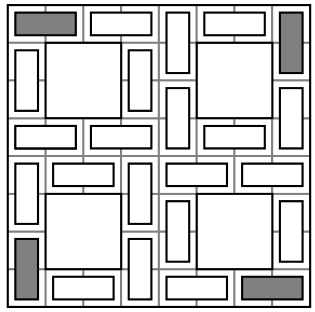
$= f_{n-1} \cdot f_{n+1}$

2. There are 4 tilings equivalent to this one:



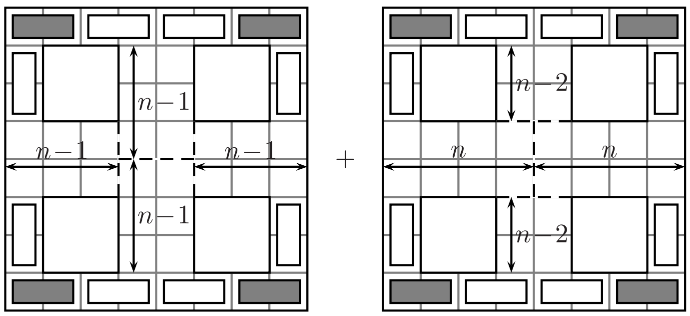
$= f_{n-1} \cdot f_{n+1}$

3. There are 2 tilings equivalent to this one:



$= 1$

4. There are 2 tilings equivalent to this one:



$= f_n^4 + f_{n-1}^2 \cdot f_{n+1}^2$

Therefore the total number of tilings is

$$\begin{aligned} T_n &= 8 \cdot f_{n-1} \cdot f_{n+1} + 4 \cdot f_{n-1} \cdot f_{n+1} + 2 + 2 \cdot (f_n^4 + f_{n-1}^2 \cdot f_{n+1}^2) \\ &= 2 \cdot f_n^4 + 2 \cdot (f_{n-1} \cdot f_{n+1})^2 + 12 \cdot f_{n-1} \cdot f_{n+1} + 2 \end{aligned}$$

which can be easily reduced to our formula by using Cassini's identity

$$f_{n+1} \cdot f_{n-1} = f_n^2 + (-1)^n.$$

□

References

[1] R. L. Graham, D. E. Knuth, O. Patashnik, *Concrete Mathematics*, Addison– Wesley, Reading MA, USA, 1989.
[2] M. Ciucu, *Enumeration of perfect matchings in graphs with reflective symmetry*, J. Combin. Theory Ser. A, **77** (1997), 67–97.
[3] N. J. A. Sloane, *The On-Line Encyclopedia of Integer Sequences*, Published electronically at http://www.research.att.com/∼njas/sequences/.