Problem 11133
(American Mathematical Monthly, Vol.112, February 2005)

Proposed by P. Bracken (USA).

Let f be a nonnegative, continuous, concave function on $[0,1]$ with $f(0)=1$. Prove that

$$2\int_0^1 x^2 f(x)\,dx + \frac{1}{12} \leq \left(\int_0^1 f(x)\,dx\right)^2.$$

Solution proposed by Roberto Tauraso, Dipartimento di Matematica, Università di Roma "Tor Vergata", via della Ricerca Scientifica, 00133 Roma, Italy.

Let $l(x)=1-x$ then, since f is a nonnegative concave function with $f(0)=1$,

$$f(x) \geq f(0) + (f(1)-f(0))x = 1 + f(1)x - x \geq l(x) \quad \text{for } x \in [0,1].$$

Moreover, it is easy to see that

$$2\int_0^1 x^2 l(x)\,dx + \frac{1}{12} = \left(\int_0^1 l(x)\,dx\right)^2.$$

Therefore, by letting $g(x)=f(x)-l(x)$, it suffices to prove

$$\begin{aligned} 2\int_0^1 x^2 g(x)\,dx &\leq \left(\int_0^1 f(x)\,dx\right)^2 - \left(\int_0^1 l(x)\,dx\right)^2 \\ &= \left(\int_0^1 g(x)\,dx\right)\left(\int_0^1 (g(x)+2l(x))\,dx\right) \\ &= \left(\int_0^1 g(x)\,dx\right)^2 + \int_0^1 g(x)\,dx, \end{aligned}$$

that is

$$\int_0^1 (2x^2-1)g(x)\,dx \leq \left(\int_0^1 g(x)\,dx\right)^2.$$

The proof is complete as soon as we show that the left-hand side is nonpositive, that is

$$\int_a^1 (2x^2-1)g(x)\,dx \leq \int_0^a (1-2x^2)g(x)\,dx$$

where $a=1/\sqrt{2}\in[0,1]$. Since g is a nonnegative continuous concave function with $g(0)=0$ then

$$g(x) \geq m'x \quad \text{for } x\in[0,a] \text{ with } m'=\tfrac{g(a)}{a}$$

and

$$g(x) \leq mx+q \quad \text{for } x \in [0,1]$$

where $y=mx+q$ is a tangent line to the graph of g at $(a,g(a))$. Now

$$\int_a^1 (2x^2-1)g(x)\,dx \leq \int_a^1 (2x^2-1)(mx+q)\,dx = \frac{m}{8} + (\sqrt{2}-1)\frac{q}{3}.$$

On the other hand,

$$\int_0^a (1-2x^2)g(x)\,dx \geq \int_0^a (1-2x^2)m'x\,dx = \frac{m'}{8}.$$

Now $m'a=g(a)=ma+q$ which implies that $m'-m=\sqrt{2}q$ and our inequality holds as soon as

$$(\sqrt{2}-1)\frac{q}{3} \leq \frac{m'-m}{8} = \frac{\sqrt{2}q}{8}$$

which is true because $q \geq g(0)=0$ and $8 > 5\sqrt{2} \approx 7.07$. $\square$