Problem 11121

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Let $k$ and $n$ be positive integers. Let $I(k, n) = \{ j \in \mathbb{N} : k^n < j < (k+1)^n \}$.
(a) For $n = 2$ and all $k$, prove that there do not exist distinct $a, b \in I(k, n)$ such that $ab$ is a square.
(b) For each $n > 2$, prove that when $k$ is sufficiently large there exist $n$ distinct integers in $I(k, n)$ whose product is the $n$th power of an integer.

Solution proposed by Roberto Tauraso, Dipartimento di Matematica, Università di Roma “Tor Vergata”, via della Ricerca Scientifica, 00133 Roma, Italy.

(a) Let $a, b \in I(k, 2)$ such that $a < b$. Assume that $ab$ is a square then there are positive integers $p < q$ and $t$ such that $a = tp^2$ and $b = tq^2$.
Since $k^2 < tp^2 < tq^2 < (k+1)^2$ then
\[
\frac{k}{p} < \sqrt{t} < \frac{k+1}{q}
\]
and
\[
q\sqrt{t} - 1 < k < p\sqrt{t}.
\]
Therefore $1 \leq q - p < 1/\sqrt{t} \leq 1$ which is a contradiction.

(b) Let $n > 2$. We first show that for $k$ is sufficiently large there is a positive integer $a_k$ such that
\[
k^n < a_k^{n-1} < (a_k + n - 2)^{n-1} < (k + 1)^n
\]
that is
\[
k^{n/(n-1)} < a_k < (k + 1)^{n/(n-1)} - n + 2.
\]
In order to prove the existence of the integer $a_k$ it suffices to show that
\[
1 < \left((k + 1)^{n/(n-1)} - n + 2\right) - k^{n/(n-1)}
\]
that is
\[
n - 1 < (k + 1)^{n/(n-1)} - k^{n/(n-1)} = k^{n/(n-1)} \left(1 + \frac{1}{k} \right)^{n/(n-1)} - 1
\]
\[
< k^{n/(n-1)} \left(1 + \frac{n}{n-1} \cdot \frac{1}{k} + O\left(\frac{1}{k^2}\right) - 1\right)
\]
\[
< \frac{n}{n-1} \cdot k^{1/(n-1)} + O\left(\frac{1}{k^{1-1/(n-1)}}\right).
\]
Since $n > 2$ the right side diverges when $k$ goes to infinity and therefore the inequality is satisfied for $k \geq k_n$ where $k_n$ is sufficiently large.
Now assume that $k \geq k_n$ and let $x_j = (a_k + j - 1)^{n-1}$ for $j = 1, \ldots, n - 1$ and $x_n = \prod_{j=1}^{n-1} (a_k + j - 1)$. Their product is a $n$th power:
\[
\prod_{j=1}^{n} x_j = \left(\prod_{j=1}^{n-1} (a_k + j - 1)\right)^n.
\]
Since
\[k^n < a_k^{n-1} = x_1 < x_2 < \cdots < x_{n-1} = (a_k + n - 2)^{n-1} < (k + 1)^n\]
then \(x_1, \ldots, x_{n-1}\) belong to \(I(k, n)\) and they are all distinct. Moreover
\[k^n < x_1 < x_n < x_{n-1} < (k + 1)^n\]
and therefore also \(x_n \in I(k, n)\). Finally \(x_n\) is different from \(x_1, \ldots, x_{n-1}\) because the product of \(n - 1\) consecutive positive integers is never a \((n - 1)\)th power.

This is a particular case of a more general result due to Erdős and Selfridge that can be found in their paper *The product of consecutive integers is never a power*, Illinois Journal of Mathematics 19 (1975), pages 292–301.

Note that when \(n\) is odd we can also take \(x_j = a_k^{n-j} \cdot (a_k + 1)^{j-1} \in I(k, n)\) for \(j = 1, \ldots, n\). Then
\[
\prod_{k=1}^{n} x_k = \left( (a_k(a_k + 1))^{(n-1)/2} \right)^n.
\]