Problem 11121
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Let k and n be positive integers. Let $I(k,n) = \{j \in \mathbb{N} : k^n < j < (k+1)^n\}$.
(a) For $n = 2$ and all k, prove that there do not exist distinct $a, b \in I(k,n)$ such that ab is a square.
(b) For each $n > 2$, prove that when k is sufficiently large there exist n distinct integers in $I(k,n)$ whose product is the nth power of an integer.

Solution proposed by Roberto Tauraso, Dipartimento di Matematica, Università di Roma "Tor Vergata", via della Ricerca Scientifica, 00133 Roma, Italy.

(a) Let $a, b \in I(k,2)$ such that $a < b$. Assume that ab is a square then there are positive integers $p < q$ and t such that $a = tp^2$ and $b = tq^2$.
Since $k^2 < tp^2 < tq^2 < (k+1)^2$ then

$$\frac{k}{p} < \sqrt{t} < \frac{k+1}{q}$$

and

$$q\sqrt{t} - 1 < k < p\sqrt{t}.$$

Therefore $1 \le q - p < 1/\sqrt{t} \le 1$ which is a contradiction.

(b) Let $n > 2$. We first show that for k is sufficiently large there is a positive integer a_k such that

$$k^n < a_k^{n-1} < (a_k + n - 2)^{n-1} < (k+1)^n$$

that is

$$k^{n/(n-1)} < a_k < (k+1)^{n/(n-1)} - n + 2.$$

In order to prove the existence of the integer a_k it suffices to show that

$$1 < \left((k+1)^{n/(n-1)} - n + 2\right) - k^{n/(n-1)}$$

that is

$$\begin{aligned} n - 1 &< (k+1)^{n/(n-1)} - k^{n/(n-1)} = k^{n/(n-1)} \left(\left(1 + \frac{1}{k}\right)^{n/(n-1)} - 1 \right) \\ &< k^{n/(n-1)} \left(1 + \frac{n}{n-1} \cdot \frac{1}{k} + O\left(\frac{1}{k^2}\right) - 1 \right) \\ &< \frac{n}{n-1} \cdot k^{1/(n-1)} + O\left(\frac{1}{k^{1-1/(n-1)}}\right). \end{aligned}$$

Since $n > 2$ the right side diverges when k goes to infinity and therefore the inequality is satisfied for $k \ge k_n$ where k_n is sufficiently large.
Now assume that $k \ge k_n$ and let $x_j = (a_k + j - 1)^{n-1}$ for $j = 1, \ldots, n-1$ and $x_n = \prod_{j=1}^{n-1} (a_k + j - 1)$.
Their product is a nth power:

$$\prod_{j=1}^{n} x_j = \left(\prod_{j=1}^{n-1} (a_k + j - 1) \right)^n.$$

Since

$$k^n < a_k^{n-1} = x_1 < x_2 < \cdots < x_{n-1} = (a_k + n - 2)^{n-1} < (k+1)^n$$

then $x_1, \ldots, x_{n-1}$ belong to $I(k,n)$ and they are all distinct. Moreover

$$k^n < x_1 < x_n < x_{n-1} < (k+1)^n$$

and therefore also $x_n \in I(k,n)$. Finally x_n is different from $x_1, \ldots, x_{n-1}$ because the product of $n-1$ consecutive positive integers is never a $(n-1)$th power. This is a particular case of a more general result due to Erdös and Selfridge that can be found in their paper *The product of consecutive integers is never a power*, Illinois Journal of Mathematics 19 (1975), pages 292–301.
Note that when n is odd we can also take $x_j = a_k^{n-j} \cdot (a_k+1)^{j-1} \in I(k,n)$ for $j = 1, \ldots, n$. Then

$$\prod_{k=1}^{n} x_k = \left((a_k(a_k+1))^{(n-1)/2} \right)^n .$$

□