Problem 11117  

Proposed by Michael Nyblom, RMIT University, Melbourne, Australia.

An integer \( n \) is a positive power if there exist integers \( a \) and \( k \) such that \( a \geq 1 \), \( m \geq 2 \), and \( n = a^m \). Let \( N(x) \) denote the number of positive powers \( n \) such that \( 1 \leq n \leq x \). For real \( x \geq 4 \) and with \( L = \lfloor \log_2 x \rfloor \), show that

\[
N(x) = \sum_{k=1}^{L-1} (-1)^{k+1} \sum_{2 \leq i_1 < \cdots < i_k \leq L} \lfloor x^{1/\text{lcm}(i_1, \ldots, i_k)} \rfloor.
\]

Solution proposed by Giulio Francot and Roberto Tauraso, Dipartimento di Matematica, Università di Roma “Tor Vergata”, via della Ricerca Scientifica, 00133 Roma, Italy.

For \( m \geq 2 \) and for \( x \geq 4 \), let \( A_m(x) \) be the set of \( m \)-powers contained in the interval \([1, x]\). Since \( |A_m(x)| = \lfloor x^{1/m} \rfloor \) then \( A_m(x) = \{1\} \) for \( m > L = \lfloor \log_2 x \rfloor \geq 2 \) and by the inclusion-exclusion principle

\[
N(x) = \left| \bigcup_{m=2}^{L} A_m(x) \right| = \sum_{k=1}^{L-1} (-1)^{k+1} \sum_{2 \leq i_1 < \cdots < i_k \leq L} |A_{i_1}(x) \cap \cdots \cap A_{i_k}(x)|
\]

Let \( p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_r^{\alpha_r} \) be the prime factorization of \( n \) then \( n \in A_m(x) \) if and only if \( 1 \leq n \leq x \) and \( m \mid \alpha_s \) for all \( s = 1, \ldots, r \). Hence

\[
A_{i_1}(x) \cap \cdots \cap A_{i_k}(x) = A_{\text{lcm}(i_1, \ldots, i_k)}(x),
\]

and therefore

\[
N(x) = \sum_{k=1}^{L-1} (-1)^{k-1} \sum_{2 \leq i_1 < \cdots < i_k \leq L} |A_{\text{lcm}(i_1, \ldots, i_k)}(x)| = \sum_{k=1}^{L-1} (-1)^{k+1} \sum_{2 \leq i_1 < \cdots < i_k \leq L} \lfloor x^{1/\text{lcm}(i_1, \ldots, i_k)} \rfloor.
\]