Problem 11117
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Proposed by M. Nyblom (Australia).

An integer n is a positive power if there exist integers a and k such that $a \geq 1$, $m \geq 2$, and $n = a^m$. Let $N(x)$ denote the number of positive powers n such that $1 \leq n \leq x$. For real $x \geq 4$ and with $L = \lfloor \log_2 x \rfloor$, show that

$$N(x) = \sum_{k=1}^{L-1} (-1)^{k+1} \sum_{2 \leq i_1 < \cdots < i_k \leq L} \lfloor x^{1/\mathrm{lcm}(i_1,\ldots,i_k)} \rfloor.$$

Solution proposed by Giulio Francot and Roberto Tauraso,
Dipartimento di Matematica, Università di Roma "Tor Vergata", via della Ricerca Scientifica, 00133 Roma, Italy.

For $m \geq 2$ and for $x \geq 4$, let $A_m(x)$ be the set of m-powers contained in the interval $[1, x]$. Since $|A_m(x)| = \lfloor x^{1/m} \rfloor$ then $A_m(x) = \{1\}$ for $m > L = \lfloor \log_2 x \rfloor \geq 2$ and by the inclusion-exclusion principle

$$N(x) = \left| \bigcup_{m=2}^{L} A_m(x) \right| = \sum_{k=1}^{L-1} (-1)^{k+1} \sum_{2 \leq i_1 < \cdots < i_k \leq L} |A_{i_1}(x) \cap \cdots \cap A_{i_k}(x)|$$

Let $p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_r^{\alpha_r}$ be the prime factorization of n then $n \in A_m(x)$ if and only if $1 \leq n \leq x$ and $m \mid \alpha_s$ for all $s = 1, \ldots, r$. Hence

$$A_{i_1}(x) \cap \cdots \cap A_{i_k}(x) = A_{\mathrm{lcm}(i_1,\ldots,i_k)}(x),$$

and therefore

$$N(x) = \sum_{k=1}^{L-1} (-1)^{k-1} \sum_{2 \leq i_1 < \cdots < i_k \leq L} \left| A_{\mathrm{lcm}(i_1,\ldots,i_k)}(x) \right| = \sum_{k=1}^{L-1} (-1)^{k+1} \sum_{2 \leq i_1 < \cdots < i_k \leq L} \lfloor x^{1/\mathrm{lcm}(i_1,\cdots,i_k)} \rfloor.$$

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