Problem 11116
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Proposed by D. Veljan (Croatia).

Let P be a convex n-gon inscribed in a circle O, and let Δ be a triangulation of P without new vertices. Compute the sum of the squares of distances from the center of O to the incenters of the triangles of Δ and show that this sum is independent of Δ.

Solution proposed by Roberto Tauraso, Dipartimento di Matematica, Università di Roma "Tor Vergata", via della Ricerca Scientifica, 00133 Roma, Italy.

Let I_k, r_k, a_k, b_k and c_k be respectively the incenter, the radius and the three sides of the kth triangle of Δ for $k = 1, \cdots, n-2$. Then, by Euler's Theorem, $|OI_k|^2 = R(R - 2r_k)$ and we have that

$$\sum_{k=1}^{n-2} |OI_k|^2 = \sum_{k=1}^{n-2} R(R - 2r_k) = (n-2)R^2 - 2R \sum_{k=1}^{n-2} r_k.$$

Moreover, by Carnot's Theorem, the signed sum of perpendicular distances from the circumcenter O to the sides of any triangle of Δ is

$$Oa_k + Ob_k + Oc_k = R + r_k.$$

Note that the sign of the distance is chosen to be positive if and only if the segment intersects the interior of the triangle and therefore, since the n-gon is convex and cyclic, we have that

$$\sum_{k=1}^{n-2} (Oa_k + Ob_k + Oc_k) = (n-2)R + \sum_{k=1}^{n-2} r_k = d(O, P)$$

where $d(O, P)$ is the signed sum of the perpendicular distances from the circumcenter O to the sides of the n-gon P, which does not depend on the triangulation. Thus also the sum of the squares of distances from the center of O to the incenters of the triangles of Δ is independent on Δ and it is equal to

$$\sum_{k=1}^{n-2} |OI_k|^2 = (n-2)R^2 - 2R(d(O,P) - (n-2)R) = R\left(3(n-2)R - 2d(O,P)\right).$$

This statement riminds a well known Sangaku problem (also called *Japanese Theorem*) which says that for this kind of triangulations the sum of the inradii is constant. □