Problem 11115  

Proposed by J. Clark (USA).  

Let \( H_n \) be the \( n \)th harmonic number, that is, \( H_n = \sum_{k=1}^{n} \frac{1}{k} \).  
Let \( E_n = H_n^2 - \sum_{k=1}^{n} \frac{1}{k} H_{\max(k,n-k)} \). Find \( \lim_{n \to \infty} E_n \).  

Solution proposed by Roberto Tauraso, Dipartimento di Matematica, Università di Roma “Tor Vergata”, via della Ricerca Scientifica, 00133 Roma, Italy.  

Let \( H_n^{(2)} = \sum_{k=1}^{n} \frac{1}{k^2} \). We will show that \( E_n = \frac{1}{2} H_n^{(2)} \lfloor \frac{n}{2} \rfloor \) and therefore \( \lim_{n \to \infty} E_n = \frac{\pi^2}{12} \).  

Since \( E_1 = \frac{1}{2} H_0^{(2)} = 0 \) and for \( n > 0 \) the difference  
\[
\frac{1}{2} H_n^{(2)} - \frac{1}{2} H_{\lfloor \frac{n}{2} \rfloor}^{(2)} = \begin{cases} 
0 & \text{if } n \text{ is even} \\
\frac{2}{(n+1)^2} & \text{if } n \text{ is odd}
\end{cases}
\]

it suffices to prove that the same holds for the difference \( E_{n+1} - E_n \). Note that  
\[
E_{n+1} - E_n = H_{n+1}^2 - \sum_{k=1}^{n+1} \frac{H_{n+1-k} - H_{n-k}}{k} - \frac{H_{n+1}}{n+1} 
\]

Assume first that \( n \) is even then \( \lfloor n/2 \rfloor = \lfloor (n+1)/2 \rfloor = n/2 \) and  
\[
E_{n+1} - E_n = H_{n+1}^2 - H_n^2 - \sum_{k=1}^{\lfloor n/2 \rfloor} \frac{H_{n+1-k} - H_{n-k}}{k} - \frac{H_{n+1}}{n+1} 
\]

\[
= \frac{H_{n+1} + H_n}{n+1} \sum_{k=1}^{\lfloor n/2 \rfloor} \frac{1}{k(n+1-k)} - \frac{H_{n+1}}{n+1} 
\]

\[
= \frac{H_n}{n+1} - \frac{1}{n+1} \sum_{k=1}^{\lfloor n/2 \rfloor} \left( \frac{1}{n+1-k} + \frac{1}{k} \right) = 0.
\]

Assume now that \( n \) is odd then \( \lfloor n/2 \rfloor = \lfloor (n+1)/2 \rfloor = (n+1)/2 \) and  
\[
E_{n+1} - E_n = H_{n+1}^2 - H_n^2 - \sum_{k=1}^{\lfloor n/2 \rfloor} \frac{H_{n+1-k} - H_{n-k}}{k} - \frac{2H_{n+1}}{n+1} - \frac{H_{n+1}}{n+1} + \frac{2H_{n+2}}{n+1} 
\]

\[
= \frac{H_{n+1} + H_n}{n+1} - \frac{1}{n+1} \sum_{k=1}^{\lfloor n/2 \rfloor} \left( \frac{1}{n+1-k} + \frac{1}{k} \right) - \frac{H_{n+1}}{n+1} 
\]

\[
= \frac{H_n}{n+1} - \frac{1}{n+1} \left( H_n - \frac{2}{n+1} \right) = \frac{2}{(n+1)^2}.
\]

\( \square \)