Problem 11115
(American Mathematical Monthly, Vol.111, November 2004)

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Let H_n be the nth harmonic number, that is, $H_n = \sum_{k=1}^n \frac{1}{k}$.
Let $E_n = H_n^2 - \sum_{k=1}^{n} \frac{1}{k} H_{\max(k,n-k)}$. Find $\lim_{n\to\infty} E_n$.

Solution proposed by Roberto Tauraso, Dipartimento di Matematica, Università di Roma "Tor Vergata", via della Ricerca Scientifica, 00133 Roma, Italy.

Let $H_n^{(2)} = \sum_{k=1}^n \frac{1}{k^2}$. We will show that

$$E_n = \frac{1}{2} H^{(2)}_{\lfloor \frac{n}{2} \rfloor}$$

and therefore

$$\lim_{n\to\infty} E_n = \frac{1}{2}\sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{12}.$$

Since $E_1 = \frac{1}{2} H_0^{(2)} = 0$ and for $n > 0$ the difference

$$\frac{1}{2} H^{(2)}_{\lfloor \frac{n+1}{2} \rfloor} - \frac{1}{2} H^{(2)}_{\lfloor \frac{n}{2} \rfloor} = \begin{cases} 0 & \text{if } n \text{ is even} \\ 2/(n+1)^2 & \text{if } n \text{ is odd} \end{cases}$$

it suffices to prove that the same holds for the difference $E_{n+1} - E_n$. Note that

$$E_n = H_n^2 - \sum_{k=1}^{\lfloor \frac{n}{2} \rfloor} \frac{H_{n-k}}{k} - \sum_{k=\lfloor \frac{n}{2} \rfloor + 1}^{n} \frac{H_k}{k}.$$

Assume first that n is even then $\lfloor n/2 \rfloor = \lfloor (n+1)/2 \rfloor = n/2$ and

$$\begin{aligned} E_{n+1} - E_n &= H_{n+1}^2 - H_n^2 - \sum_{k=1}^{\frac{n}{2}} \frac{H_{n+1-k} - H_{n-k}}{k} - \frac{H_{n+1}}{n+1} \\ &= \frac{H_{n+1} + H_n}{n+1} - \sum_{k=1}^{\frac{n}{2}} \frac{1}{k(n+1-k)} - \frac{H_{n+1}}{n+1} \\ &= \frac{H_n}{n+1} - \frac{1}{n+1} \sum_{k=1}^{\frac{n}{2}} \left(\frac{1}{n+1-k} + \frac{1}{k} \right) = 0. \end{aligned}$$

Assume now that n is odd then $\lfloor n/2 \rfloor = (n-1)/2$, $\lfloor (n+1)/2 \rfloor = (n+1)/2$ and

$$\begin{aligned} E_{n+1} - E_n &= H_{n+1}^2 - H_n^2 - \sum_{k=1}^{\frac{n-1}{2}} \frac{H_{n+1-k} - H_{n-k}}{k} - \frac{2H_{\frac{n+1}{2}}}{n+1} - \frac{H_{n+1}}{n+1} + \frac{2H_{\frac{n+1}{2}}}{n+1} \\ &= \frac{H_{n+1} + H_n}{n+1} - \frac{1}{n+1} \sum_{k=1}^{\frac{n-1}{2}} \left(\frac{1}{n+1-k} + \frac{1}{k} \right) - \frac{H_{n+1}}{n+1} \\ &= \frac{H_n}{n+1} - \frac{1}{n+1} \left(H_n - \frac{2}{n+1} \right) = \frac{2}{(n+1)^2}. \end{aligned}$$

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