Problem 11103
(American Mathematical Monthly, Vol.111, October 2004)

Proposed by G. Galperin and H. Gauchman (USA).

Prove that for every positive integer n,

$$\sum_{k=1}^{n} \frac{1}{k\binom{n}{k}} = \frac{1}{2^{n-1}} \sum_{\substack{k=1 \\ k \text{ odd}}}^{n} \frac{\binom{n}{k}}{k}.$$

Solution proposed by Roberto Tauraso, Dipartimento di Matematica, Università di Roma "Tor Vergata", via della Ricerca Scientifica, 00133 Roma, Italy.

We first note that

$$\sum_{k=1}^{n} \frac{1}{k\binom{n}{k}} = \sum_{k=1}^{n} \frac{1}{n\binom{n-1}{k-1}} = \frac{1}{n}\sum_{k=0}^{n-1} \frac{1}{\binom{n-1}{k}} = \frac{1}{2^n} \cdot \sum_{k=0}^{n-1} \frac{2^{k+1}}{k+1}.$$

We prove the last equality by induction (see also Problem 1682 of the MAGAZINE). It holds for $n = 1$. Now take $n > 1$ and note that for $k = 0, \ldots, n-2$

$$\frac{1}{\binom{n-1}{k+1}} + \frac{1}{\binom{n-1}{k}} = \frac{\binom{n}{k+1}}{\binom{n-1}{k+1} \cdot \binom{n-1}{k}} = \frac{\frac{n}{k+1} \cdot \binom{n-1}{k}}{\frac{n-1}{k+1} \cdot \binom{n-2}{k} \cdot \binom{n-1}{k}} = \frac{n}{n-1} \cdot \frac{1}{\binom{n-2}{k}}.$$

Then summing over k and using the induction hypothesis we get

$$\sum_{k=0}^{n-2} \frac{1}{\binom{n-1}{k+1}} + \sum_{k=0}^{n-2} \frac{1}{\binom{n-1}{k}} = \frac{n}{n-1} \cdot \sum_{k=0}^{n-2} \frac{1}{\binom{n-2}{k}} = \frac{n}{2^{n-1}} \cdot \sum_{k=0}^{n-2} \frac{2^{k+1}}{k+1}$$

that is

$$\sum_{k=1}^{n-1} \frac{1}{\binom{n-1}{k}} + \sum_{k=0}^{n-2} \frac{1}{\binom{n-1}{k}} = 2\sum_{k=0}^{n-1} \frac{1}{\binom{n-1}{k}} - 1 - 1 = \frac{n}{2^{n-1}} \cdot \sum_{k=0}^{n-2} \frac{2^{k+1}}{k+1}$$

and finally

$$\frac{1}{n}\sum_{k=0}^{n-1} \frac{1}{\binom{n-1}{k}} = \frac{1}{2^n} \cdot \sum_{k=0}^{n-2} \frac{2^{k+1}}{k+1} + \frac{1}{n} = \frac{1}{2^n} \cdot \sum_{k=0}^{n-1} \frac{2^{k+1}}{k+1}.$$

On the other hand

$$\frac{1}{2^{n-1}} \cdot \sum_{\substack{k=1 \\ k \text{ odd}}}^{n} \frac{\binom{n}{k}}{k} = \frac{1}{2^n}\sum_{k=1}^{n} \binom{n}{k} \int_{-1}^{1} x^{k-1}\, dx = \frac{1}{2^n}\int_{-1}^{1} \frac{(1+x)^n - 1}{x}\, dx$$

$$= \frac{1}{2^n}\int_{0}^{2} \frac{y^n - 1}{y-1}\, dy = \frac{1}{2^n}\sum_{k=0}^{n-1}\int_{0}^{2} y^k\, dy = \frac{1}{2^n} \cdot \sum_{k=0}^{n-1} \frac{2^{k+1}}{k+1}.$$

Hence

$$\sum_{k=1}^{n} \frac{1}{k\binom{n}{k}} = \frac{1}{2^n} \cdot \sum_{k=0}^{n-1} \frac{2^{k+1}}{k+1} = \frac{1}{2^{n-1}} \sum_{\substack{k=1 \\ k \text{ odd}}}^{n} \frac{\binom{n}{k}}{k}.$$

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