Problem 11098
(American Mathematical Monthly, Vol.111, August-September 2004)

Proposed by C. Hillar and D. Rhea (USA).

Let

$$f(n)=\sum_{i=1}^{n}\frac{(-1)^{i+1}}{2^i-1}\binom{n}{i}.$$

Prove that there are constants c and c′ such that $c\leq f(n)/\log n\leq c'$ for sufficiently large n (that is, $f(n)=\Theta(\log n)$).

Solution proposed by Roberto Tauraso, Dipartimento di Matematica, Università di Roma "Tor Vergata", via della Ricerca Scientifica, 00133 Roma, Italy.

For $n\geq 1$ consider the difference (we can assume that $f(0)=0$)

$$\begin{aligned} f(n)-f(n-1) &= \sum_{i=1}^{n}\frac{(-1)^{i+1}}{2^i-1}\left[\binom{n}{i}-\binom{n-1}{i}\right]\\ &= \sum_{i=1}^{n}(-1)^{i+1}\left[\sum_{k=1}^{\infty}\left(\frac{1}{2^i}\right)^k\right]\binom{n-1}{i-1}\\ &= \sum_{i=1}^{n}(-1)^{i+1}\left[\sum_{k=1}^{\infty}\left(\frac{1}{2^k}\right)^i\right]\binom{n-1}{i-1}\\ &= \sum_{k=1}^{\infty}\frac{1}{2^k}\left[\sum_{i=1}^{n}\binom{n-1}{i-1}\left(-\frac{1}{2^k}\right)^{i-1}\right]\\ &= \sum_{k=1}^{\infty}\frac{1}{2^k}\left[\sum_{i=0}^{n-1}\binom{n-1}{i}\left(-\frac{1}{2^k}\right)^{i}\right]=\sum_{k=1}^{\infty}\left(\frac{1}{2^k}\right)\left(1-\frac{1}{2^k}\right)^{n-1}. \end{aligned}$$

Let $x_k=\sum_{j=1}^{k}1/2^j=1-1/2^k$ for $k\geq 0$ then

$$f(n)-f(n-1)=\sum_{k=1}^{\infty}(x_k-x_{k-1})x_k^{n-1}=\int_0^1\varphi(x)\,dx\geq\int_0^1 x^{n-1}\,dx=\frac{1}{n}$$

because x^{n-1} is increasing in $[0,1]$ and

$$\varphi(x)=\sum_{k=1}^{\infty}x_k^{n-1}\,\chi_{[x_{k-1},x_k]}(x)\geq x^{n-1}\quad\text{for } x\in[0,1].$$

Since $x_k-x_{k-1}=2(x_{k+1}-x_k)$, we have also that

$$f(n)-f(n-1)=2\sum_{k=1}^{\infty}(x_{k+1}-x_k)x_k^{n-1}=2\int_0^1\psi(x)\,dx\leq 2\int_0^1 x^{n-1}\,dx=\frac{2}{n}$$

where

$$\psi(x)=\sum_{k=1}^{\infty}x_k^{n-1}\,\chi_{[x_k,x_{k+1}]}(x)\leq x^{n-1}\quad\text{for } x\in[0,1].$$

Therefore

$$H_n=\sum_{i=1}^{n}\frac{1}{i}\leq f(n)=\sum_{i=1}^{n}(f(i)-f(i-1))+f(0)\leq 2\sum_{i=1}^{n}\frac{1}{i}=2H_n$$

and since $H_n=\Theta(\log n)$ we have also that $f(n)=\Theta(\log n)$. $\square$