Problem 11091
(American Mathematical Monthly, Vol.111, June-July 2004)

Proposed by D. Callan (USA).

Let $x_0, x_1, x_2, \ldots, x_n$ be indeterminates. It is well known that the number of ways to insert redundant parentheses into the repeated quotient $x_0 \div x_1 \div x_2 \div \cdots \div x_n$ to make it meaningful expression is the nth Catalan number, C_n.

The resulting fractions are not all the same, because division is not associative, but neither are they all different. Which fraction occurs most frequently, and how often does it occur? (The Catalan sequence begins with 1, 1, 2, 5, 14, 42, 132 *and* 429.*)*

Solution proposed by Roberto Tauraso, Dipartimento di Matematica, Università di Roma "Tor Vergata", via della Ricerca Scientifica, 00133 Roma, Italy.

Given a quotient with parentheses, we consider the sequence $p_1, p_2, \ldots, p_n$ where p_i is the number of right parentheses which are on the left of x_i. Then x_i for $i = 1, \ldots n$ is at the numerator or at the denominator of the fraction whether i, the number of divisions on the left of x_i, minus p_i is even or odd (note that x_0 is always at the numerator). Actually the sequences of integers $0 \leq p_1 \leq p_2 \leq \cdots \leq p_n$ such that $p_i < i$ encode in a unique way the fractions with parentheses.
Hence two fractions with parentheses are equivalent if and only if their corresponding sequences are equal modulo 2. Moreover the most frequently fraction is given by the sequences $p_1, p_2, \ldots, p_n$ where p_i is always even, that is the quotient of the product of the x_i's with i even and the product of the x_i's with i odd. We denote by A_n the number of times that this fraction occurs. For example if $n = 3$ then $C_3 = 5$ and $A_3 = 2$:

Quotient with parentheses	$p_1\, p_2\, p_3$	Fraction
$(x_0 \div (x_1 \div (x_2 \div x_3)))$	0 0 0	$(\mathbf{x_0 \cdot x_2}) \div (\mathbf{x_1 \cdot x_3})$
$(x_0 \div ((x_1 \div x_2) \div x_3)$	0 0 1	$(x_0 \cdot x_2 \cdot x_3) \div (x_1)$
$(x_0 \div (x_1 \div x_2)) \div x_3)$	0 0 2	$(\mathbf{x_0 \cdot x_2}) \div (\mathbf{x_1 \cdot x_3})$
$((x_0 \div x_1) \div (x_2 \div x_3))$	0 1 1	$(x_0 \cdot x_3) \div (x_1 \cdot x_2)$
$(((x_0 \div x_1) \div x_2) \div x_3)$	0 1 2	$(x_0) \div (x_1 \cdot x_2 \cdot x_3)$

It is easy to construct a bijective correspondence between the sequences $p_1, p_2, \ldots, p_n$ where p_i is always even, and the number of 3-good paths introduced by P. Hilton and J. Pederson in *Catalan Numbers, Their Generalization, and Their Uses*, Math. Intel. 13, 64–75, 1991. More precisely, if $n = 2k$ then A_n is the number of lattice paths from $(0, -1)$ to $(k, 2k-1)$ entirely below the line $y = 2x$ and the formula is

$$A_n = \frac{1}{2k+1}\binom{3k}{k}.$$

On the other hand, if $n = 2k+1$ then A_n is the number of lattice paths from $(1, 0)$ to $(k+1, 2k+2)$ entirely below the line $y = 2x$ and the formula is

$$A_n = \frac{1}{2k+1}\binom{3k+1}{k+1}.$$

Here there are some values of C_n and of A_n:

n	1	2	3	4	5	6	7
C_n	1	2	5	14	42	132	439
A_n	1	1	2	3	7	12	30

□