Problem 11078
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Proposed by D. Knuth (USA).

A positive integer is cube-free if it is not divisible by the cube of any integer greater than 1. Let $\sum^*$ denote the summation restricted to the cube-free integers.
(**a**) *Evaluate $\sum^* n^{-2}$.*
(**b**) *Prove that the sum $\sum^*_{n \text{ odd}}(-1)^{(n-1)/2}\, n^{-1}$ converges and determine its value.*

Solution proposed by Roberto Tauraso, Dipartimento di Matematica, Università di Roma "Tor Vergata", via della Ricerca Scientifica, 00133 Roma, Italy.

(**a**) Let N be an integer greater than 1 and define the following Dirichlet series for $\mathrm{Re}(s) > 1$:

$$F_N(s) = \sum_{n=1}^{\infty} \frac{f_N(n)}{n^s} \quad \text{with } f_N(n) = [n \text{ is a } N\text{th-power free integer}],$$

$$G_N(s) = \sum_{n=1}^{\infty} \frac{g_N(n)}{n^s} = \zeta(Ns) \quad \text{with } g_N(n) = [n \text{ is a } N\text{th-power integer}].$$

The Dirichlet convolution of f_N and g_N is

$$(f_N * g_N)(n) = \sum_{d|n} f_N(d) g_N(n/d) = 1$$

because there is a unique way to write an integer n as the product of a Nth-power free integer by a Nth-power integer. Therefore for $\mathrm{Re}(s) > 1$

$$F_N(s) G_N(s) = \sum_{n=1}^{\infty} \frac{(f_N * g_N)(n)}{n^s} \sum_{n=1}^{\infty} \frac{1}{n^s} = \zeta(s)$$

and

$$F_N(s) = \frac{\zeta(s)}{G_N(s)} = \frac{\zeta(s)}{\zeta(Ns)}.$$

In particular for $s = 2$ and $N = 3$ we obtain

$$\sum^* \frac{1}{n^2} = F_3(2) = \frac{\zeta(2)}{\zeta(6)} = \frac{\pi^2/6}{\pi^6/945} = \frac{315}{2\pi^4}.$$

(**b**) For $n \geq 1$ the arithmetical function

$$\chi(n) = \begin{cases} (-1)^{(n-1)/2} & \text{if } n \text{ is odd,} \\ 0 & \text{if } n \text{ is even} \end{cases}$$

is completely multiplicative and

$$\begin{aligned} (\chi \cdot f_N) * (\chi \cdot g_N)(n) &= \sum_{d|n} \chi(d) f_N(d) \chi(n/d) g_N(n/d) \\ &= \chi(n) \sum_{d|n} f_N(d) g_N(n/d) = \chi(n)(f_N * g_N)(n). \end{aligned}$$

For $\mathrm{Re}(s) > 1$, let

$$\tilde{F}_N(s) = \sum_{n=1}^{\infty} \frac{\chi(n) f_N(n)}{n^s},$$

$$\tilde{G}_N(s) = \sum_{n=1}^{\infty} \frac{\chi(n) g_N(n)}{n^s} = \sum_{n=1}^{\infty} \frac{\chi(n^N)}{n^{Ns}} = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^{Ns}} = \sum_{n \text{ odd}} \frac{(-1)^{(n-1)/2}}{n^{Ns}}$$

then

$$\tilde{H}(s) = \tilde{F}_N(s)\tilde{G}_N(s) = \sum_{n=1}^{\infty} \frac{\chi(n)(f_N * g_N)(n)}{n^s} = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s} = \sum_{n \text{ odd}} \frac{(-1)^{(n-1)/2}}{n^s}.$$

Since $\chi(1)g_N(1) = 1 \neq 0$, then $1/\tilde{G}_N(s)$ is a Dirichlet series which converges absolutely for $\mathrm{Re}(s) > 1/N$:

$$1/\tilde{G}_N(s) = \sum_{n=1}^{\infty} \frac{\mu(n)\chi(n)}{n^{Ns}} = \sum_{n=1}^{\infty} \frac{\mu(n^{1/N})\chi(n)g_N(n)}{n^s}.$$

Moreover, the series $\tilde{H}(s)$ converges conditionally in the strip $0 < \mathrm{Re}(s) \leq 1$ and absolutely for $\mathrm{Re}(s) > 1$, therefore the product of $\tilde{H}(s)$ and $1/\tilde{G}_N(s)$ is a Dirichlet series which converges not only for $\mathrm{Re}(s) > 1$ but also for $s = 1$ (this is the analogous of Mertens' Theorem for Dirichlet series). Then by Uniqueness Theorem

$$\chi(n)f_N(n) = \chi(n) * \mu(n^{1/N})\chi(n)g_N(n) = \chi(n) \sum_{d^N | n} \mu(n)$$

and the identity

$$\tilde{F}_N(s) = \tilde{H}(s) \cdot 1/\tilde{G}_N(s)$$

holds also for $s = 1$. In particular for $N = 3$ we obtain

$$\sum_{n \text{ odd}}^{*} \frac{(-1)^{(n-1)/2}}{n} = \tilde{F}_3(1) = \frac{\sum_{n \text{ odd}} (-1)^{(n-1)/2}\, n^{-1}}{\sum_{n \text{ odd}} (-1)^{(n-1)/2}\, n^{-3}} = \frac{\pi/4}{\pi^3/32} = \frac{8}{\pi^2}.$$

□