Problem 11075
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Proposed by G. Trenkler (Germany).

Let a, b, and c be complex numbers. Show that

$$\left|\sqrt{a^2+b^2+c^2}\right| \leq \max\left\{|a|+|b|, |b|+|c|, |a|+|c|\right\}.$$

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We consider the complex anti-symmetric matrix

$$M = \begin{pmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{pmatrix}$$

whose eigenvalues are $0, \pm\sqrt{a^2+b^2+c^2}$. Then the inequality follows from the Gerschgorin Theorem. However we give here a direct proof.
Let $\lambda = \sqrt{a^2+b^2+c^2}$ (one of the squareroots) and let $x = (x_1, x_2, x_3) \neq 0$ be a corresponding eigenvector. Since $Mx = \lambda x$ then

$$\begin{cases} ax_2 + bx_3 = \lambda x_1 \\ -ax_1 + cx_3 = \lambda x_2 \\ -bx_1 - cx_2 = \lambda x_3 \end{cases}$$

and

$$\begin{cases} |\lambda||x_1| \leq |a||x_2| + |b||x_3| \leq (|a|+|b|) \cdot \max\limits_{i=2,3} |x_i| \leq R \cdot \max\limits_{i=1,2,3} |x_i| \\ |\lambda||x_2| \leq |a||x_1| + |c||x_3| \leq (|a|+|c|) \cdot \max\limits_{i=1,3} |x_i| \leq R \cdot \max\limits_{i=1,2,3} |x_i| \\ |\lambda||x_3| \leq |b||x_1| + |c||x_2| \leq (|b|+|c|) \cdot \max\limits_{i=1,2} |x_i| \leq R \cdot \max\limits_{i=1,2,3} |x_i| \end{cases} \tag{1}$$

where $R = \max\{|a|+|b|, |b|+|c|, |a|+|c|\}$. Therefore

$$|\lambda| \cdot \max_{i=1,2,3} |x_i| \leq R \cdot \max_{i=1,2,3} |x_i|.$$

Since $\max\limits_{i=1,2,3} |x_i| > 0$ then

$$\left|\sqrt{a^2+b^2+c^2}\right| = \lambda \leq R = \{|a|+|b|, |b|+|c|, |a|+|c|\}.$$

Note that the equality holds if and only if at least two of the numbers a, b and c are zero. If the is only one zero, say c, then

$$\lambda = \left|\sqrt{a^2+b^2}\right| = \sqrt{|a^2+b^2|} \leq \sqrt{|a|^2+|b|^2} < |a|+|b| = R.$$

Now assume that a, b and c are all different from zero and that $\lambda = R$. Then all the inequalities in (**??**) become equalities and it follows that

$$|x_1| = |x_2| = |x_3| \neq 0 \quad \text{and} \quad |a|+|b| = |b|+|c| = |a|+|c| = R,$$

that is $|a| = |b| = |c| \neq 0$ which yields the contradiction

$$\sqrt{3}|a| = \lambda = R = 2|a|.$$

$\square$