Problem 11070
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Let f and g be two commuting analytic maps from a non-empty open connected set $D \subset \mathbb{C}$ into D. Suppose that $z_0 \in D$ be a fixed point of both f and g, and that neither $f'(z_0)$ nor $g'(z_0)$ is a root of unity. Suppose also there exists an integer $N \geq 1$ such that $f^{(k)}(z_0) = g^{(k)}(z_0) = 0$ for $1 \leq k \leq N-1$, while $f^{(N)}(z_0) = g^{(N)}(z_0) \neq 0$. Prove that the restriction of f and g to D are equal.

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We will use the Faà di Bruno's formula for the nth-derivative of the composition of two functions: letting $f_n = f^{(n)}(z_0)$ and $g_n = g^{(n)}(z_0)$ for $n \geq 1$ then

$$(f \circ g)^{(n)}(z_0) = B_n(f_1, g_1, \cdots, f_n, g_n)$$

where B_n is the Bell polynomial of degree n

$$B_n(x_1, y_1, \cdots, x_n, y_n) = \sum_{\pi(n)} \frac{n!}{r_1!r_2!\cdots r_n!(1!)^{r_1}(2!)^{r_2}\cdots(n!)^{r_n}} \cdot x_r y_1^{r_1} y_2^{r_2} \cdots y_n^{r_n}.$$

The sum runs over all partitions $\pi(n)$ of the integer n: $n = r_1 + 2r_2 + \cdots + nr_n$, r_k denotes the number of parts of size k, and $r = r_1 + r_2 + \cdots + r_n$ is the total number of parts.
By hypothesis $f_k = g_k$ for $k = 1, \ldots, N$ and deriving the identity $f \circ g = g \circ f$ we get useful relations that will help us to prove by induction that $f_n = g_n$ for all n that is $f \equiv g$:

$$(f \circ g)^{(n)}(z_0) = B_n(f_1, g_1, \cdots, f_n, g_n) = B_n(g_1, f_1, \cdots, g_n, f_n) = (g \circ f)^{(n)}(z_0).$$

First we consider the case $N = 1$ and assume that $f_k = g_k$ for $k = 1, \ldots, n-1$ and $n \geq 2$. It easy to see that the monomials in $B_n(f_1, g_1, \cdots, f_n, g_n)$ which depend on f_n or g_n are precisely

$$f_1 g_n \quad \text{and} \quad f_n g_1^n.$$

Gathering all the other monomials in the polynomial $P_n(f_1, g_1, \cdots, f_{n-1}, g_{n-1})$ we have that

$$(f \circ g)^{(n)}(z_0) = f_1 g_n + P_n(f_1, g_1, \cdots, f_{n-1}, g_{n-1}) + f_n g_1^n$$

and similarly by swapping f and g we obtain

$$(g \circ f)^{(n)}(z_0) = g_1 f_n + P_n(g_1, f_1, \cdots, g_{n-1}, f_{n-1}) + g_n f_1^n.$$

Therefore

$$f_1 g_n + P_n(f_1, g_1, \cdots, f_{n-1}, g_{n-1}) + f_n g_1^n = g_1 f_n + P_n(g_1, f_1, \cdots, g_{n-1}, f_{n-1}) + g_n f_1^n,$$

and since $f_k = g_k$ for $k = 1, \ldots, n-1$ we get

$$g_1(g_1^{n-1} - 1) f_n = f_1(f_1^{n-1} - 1) g_n$$

and we conclude that $f_n = g_n$ because $f_1 = g_1 \neq 0$ is not a root of unity.
Now suppose that $N > 1$ and assume that $f_k = g_k$ for $k = 1, \ldots, N+m-1$ with $m \geq 1$. Let's consider the Bell polynomial $B_n(f_1, g_1, \cdots, f_n, g_n)$ for $n = N^2 + m$. Since $f_k = g_k = 0$ for $1 \leq k \leq N-1$ it suffices to take into account the monomials

$$f_r g_1^{r_1} g_2^{r_2} \cdots g_n^{r_n}$$

with $r_1 = r_2 = \cdots = r_{N-1} = 0$ and $r \geq N$. Then

$$\sum_{k=1}^{n} r_k = \sum_{k=N}^{n} r_k = r \quad \text{and} \quad \sum_{k=1}^{n} kr_k = \sum_{k=N}^{n} kr_k = n.$$

and by subtracting N times the first sum from the second one we get

$$0 \leq \sum_{k=N}^{n} (k-N) r_k = \sum_{k=1}^{n-N} kr_{N+k} = n - rN \leq n - N^2 = m$$

This implies that $r_{N+k} = 0$ for $k > m$ and $r \leq n/N = N + m/N < N + m$. Moreover, if $r_{N+m} \neq 0$ then $r_{N+m} = 1$, $r_{N+k} = 0$ for $k = 1, \cdots, m-1$, $r_N = N-1$, and $r = r_N + r_{N+m} = N$. Thus we have selected the monomial

$$f_N g_N^{N-1} g_{N+m}.$$

On the other hand, if $r_{N+m} = 0$ then the corresponding monomials depend on f_k and g_k for $k = N, \cdots, N+m-1$. We gather these monomials in the polynomial $Q_n(f_N, g_N, \cdots, f_{N+m-1}, g_{N+m-1})$. Thus

$$(f \circ g)^{(n)}(z_0) = c f_N g_N^{N-1} g_{N+m} + Q_n(f_N, g_N, \cdots, f_{N+m-1}, g_{N+m-1})$$

and in the same way

$$(g \circ f)^{(n)}(z_0) = c g_N f_N^{N-1} f_{N+m} + Q_n(g_N, f_N, \cdots, g_{N+m-1}, f_{N+m-1})$$

where c is a positive coefficient. Since $f_k = g_k$ for $k = 1, \ldots, N+m-1$, after equating we get

$$f_N g_N^{N-1} g_{N+m} = g_N f_N^{N-1} f_{N+m}$$

and we conclude that $f_{N+m} = g_{N+m}$ because $f_N = g_N \neq 0$. $\square$

Remark. The condition that the first derivatives are not roots of unity is necessary: the maps $f(z) = z^{n+1} + \omega z$ and $g(z) = \omega z$ with $\omega^n = 1$ commute, $f(0) = g(0) = 0$ and $f'(0) = g'(0) = \omega \neq 0$ but $f \not\equiv g$

If $|f'(z_0)| = |g'(z_0)| \neq 1$ then the statement can be proved also by the existence of a common local conjugation due to Schröder and Böttcher theorems. This approach is much more difficult when z_0 is an irrationally indifferent fixed point.