Problem 11068
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Proposed by H. Wilf (USA).

For a rational number x that equals a/b in lowest terms, let $f(x) = ab$.
(**a**) *Show that*

$$\sum_{x\in\mathbb{Q}^+} \frac{1}{f(x)^2} = \frac{5}{2},$$

where the sum extends over all positive rationals.
(**b**) *More generally, exhibit an infinite sequence of distinct rational exponents s such that $\sum_{x\in\mathbb{Q}^+} f(x)^{-s}$ is rational.*

Solution proposed by Roberto Tauraso, Dipartimento di Matematica, Università di Roma "Tor Vergata", via della Ricerca Scientifica, 00133 Roma, Italy.

We first note that

$$F(s) = \sum_{x\in\mathbb{Q}^+} \frac{1}{f(x)^s} = \sum_{\substack{a,\,b\,=\,1,\\ \gcd(a,b)\,=\,1}}^{\infty} \frac{1}{(a\cdot b)^s}.$$

Moreover for $s > 1$ we have that

$$\zeta(s)^2 = \left(\sum_{a=1}^{\infty} \frac{1}{a^s}\right)^2 = \sum_{a,b=1}^{\infty} \frac{1}{(a\cdot b)^s} = \sum_{d=1}^{\infty} \sum_{\substack{a,\,b\,=\,1,\\ \gcd(a,b)\,=\,d}}^{\infty} \frac{1}{(a\cdot b)^s}$$

$$= \sum_{d=1}^{\infty} \frac{1}{d^{2s}} \cdot \sum_{\substack{a,\,b\,=\,1,\\ \gcd(a,b)\,=\,1}}^{\infty} \frac{1}{(a\cdot b)^s} = \zeta(2s)\cdot F(s).$$

Therefore if s is a positive even number $2n$ then the sum converges to

$$\begin{aligned} F(2n) &= \frac{\zeta(2n)^2}{\zeta(4n)} = \left((-1)^{n-1}\frac{2^{2n-1}B_{2n}\pi^{2n}}{(2n)!}\right)^2 \left((-1)^{2n-1}\frac{2^{4n-1}B_{4n}\pi^{4n}}{(4n)!}\right) \\ &= \binom{4n}{2n}\cdot\frac{B_{2n}^2}{2|B_{4n}|} \in \mathbb{Q} \end{aligned}$$

where B_k is the k-th Bernoulli number (it is rational!). Here there are some values of the function $F(2n)$:

$$F(2) = \frac{5}{2}\,, \quad F(4) = \frac{7}{6}\,, \quad F(6) = \frac{715}{691}\,, \quad F(8) = \frac{7293}{7234}.$$

$\square$