Problem 11068

Proposed by Herbert Wilf, University of Pennsylvania, Philadelphia, PA.

For a rational number \( x \) that equals \( a/b \) in lowest terms, let \( f(x) = ab \).

(a) Show that
\[
\sum_{x \in \mathbb{Q}^+} \frac{1}{f(x)^2} = \frac{5}{2},
\]
where the sum extends over all positive rationals.

(b) More generally, exhibit an infinite sequence of distinct rational exponents \( s \) such that \( \sum_{x \in \mathbb{Q}^+} f(x)^{-s} \) is rational.

Solution proposed by Roberto Tauraso, Dipartimento di Matematica, Università di Roma “Tor Vergata”, via della Ricerca Scientifica, 00133 Roma, Italy.

We first note that
\[
F(s) = \sum_{x \in \mathbb{Q}^+} \frac{1}{f(x)^s} = \sum_{a,b = 1}^{\infty} \frac{1}{(a \cdot b)^s}.
\]

Moreover for \( s > 1 \) we have that
\[
\zeta(s)^2 = \left( \sum_{a=1}^{\infty} \frac{1}{a^s} \right)^2 = \sum_{a,b = 1}^{\infty} \frac{1}{(a \cdot b)^s} \sum_{d \mid \gcd(a,b)} \frac{1}{d^{2s}}
\]
\[
= \sum_{d=1}^{\infty} \frac{1}{d^{2s}} \cdot \sum_{a,b = 1}^{\infty} \frac{1}{(a \cdot b)^s} = \zeta(2s) \cdot F(s).
\]

Therefore if \( s \) is a positive even number \( 2n \) then the sum converges to
\[
F(2n) = \frac{\zeta(2n)^2}{\zeta(4n)} = \left( (-1)^{n-1} \frac{2^{2n-1} B_{2n} \pi^{2n}}{(2n)!} \right)^2 \left( (-1)^{2n-1} \frac{2^{4n-1} B_{4n} \pi^{4n}}{(4n)!} \right)
\]
\[
= \left( \frac{4n}{2n} \right) \cdot \frac{B_{2n}^2}{2|B_{4n}|} \in \mathbb{Q}
\]
where \( B_k \) is the \( k \)-th Bernoulli number (it is rational!). Here there are some values of the function \( F(2n) \):
\[
F(2) = \frac{5}{2}, \quad F(4) = \frac{7}{6}, \quad F(6) = \frac{715}{691}, \quad F(8) = \frac{7293}{7234}.
\]

\( \square \)