Problem 11060
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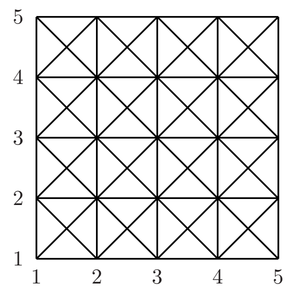
Proposed by M. Coppenbarger (USA).

Let $[n]$ denote the set of integers $\{1, 2, \ldots, n\}$. Let G_n be the union of all closed line segments joining any two elements of $[n] \times [n]$ along a vertical or horizontal line, or along a line with slope ± 1. Determine the combined total F_n of the number of (nondegenerate) triangles and rectangles for which all edges are subsets of G_n. (The vertices of these figures need not to be in $[n] \times [n]$).

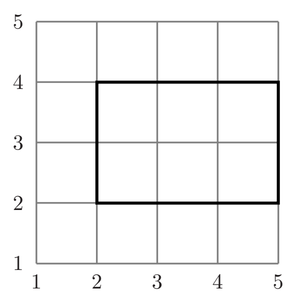
Solution proposed by Roberto Tauraso, Dipartimento di Matematica, Università di Roma "Tor Vergata", via della Ricerca Scientifica, 00133 Roma, Italy.

Here is the set G_n for $n = 5$:



In order to compute the combined total F_n of the number of triangles and rectangles for which all edges are subsets of G_n, we consider four cases.

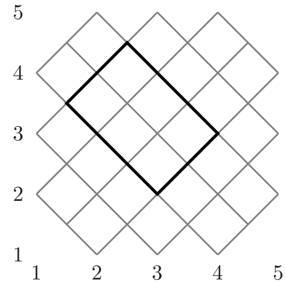
(**1**) We first count the rectangles with horizontal and vertical sides:



To determine a rectangle of this type we have to choose two different horizontal lines and two different vertical lines. Therefore the total number of rectangles of this type is

$$R_{1,n} = \binom{n}{2} \cdot \binom{n}{2} = \frac{1}{4}(n^4 - 2n + n^2).$$

(**2**) Now we count the rectangles with sides of slopes ± 1:



The above picture is the union of $n-2$ rectangles of size $[2n-(2k+1)]\times[2k+1]$ for $k=1,\dots,n-2$ which are centered at the same point. The number of rectangles with all four vertices in $\bigcup_{k=1}^{s}[2n-(2k+1)]\times[2k+1]$ for $s=1,\dots,n-2$ is

$$R_{2,n}(s)=\sum_{k=1}^{s}\binom{2n-(2k+1)}{2}\left(\binom{2k+1}{2}-\binom{2k-1}{2}\right).$$

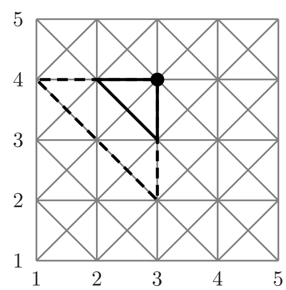
By the Inclusion-Exclusion Principle, it suffices to note that for $s>1$ the difference $R_{2,n}(s)-R_{2,n}(s-1)$ is the number of rectangles with all four vertices in $[2n-(2s+1)]\times[2s+1]$ minus the number of rectangles with all four vertices in the intersection

$$[2n-(2s+1)]\times[2s+1]\cap\bigcup_{k=1}^{s-1}[2n-(2k+1)]\times[2k+1]=[2n-(2s+1)]\times[2s-1].$$

Therefore the total number of rectangles of this type is

$$R_{2,n}=R_{2,n}(n-2)=\frac{1}{6}(4n^4-16n^3+23n^2-17n+6).$$

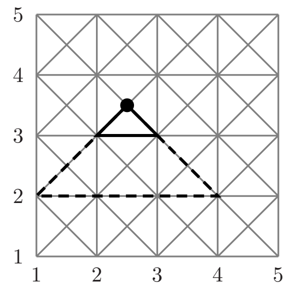
(**3**) We consider the triangles with a horizontal side, a vertical side and the last one of slope ± 1. There are four possible orientations and by simmetry it suffices to consider only one:



The number of triangles of this type with the right angle at a given point $(i,j)\in[n]\times[n]$ and oriented as the one in the picture is $\min(i-1,j-1)$. Therefore the total number of triangles of this type is

$$T_{1,n}=4\cdot\sum_{i=1}^{n}\sum_{j=1}^{n}\min(i-1,j-1)=\frac{1}{3}(4n^3-6n^2+2n).$$

(**4**) Finally we count the triangles with a side of slope 1, another side of slope -1 and the last one horizontal or vertical. As before, there are four possible orientations and by simmetry it suffices to consider only one:



The number of triangles of this type with the right angle at a given point$(i,j) \in [n]\times[n]$ and oriented as the one in the picture is $\min(j-1, n-j, i-1)$. Moreover, in this case there are also triangles with the right angle at $(1/2+i, 1/2+j)$ for $(i,j) \in [n-1]\times[n-1]$ and their number is $\min(j, n-j, i)$. Therefore the total number of triangles of this type is

$$\begin{aligned} T_{n,2} &= 4\cdot\left(\sum_{i=1}^{n}\sum_{j=1}^{n}\min(j-1,n-j,i-1)+\sum_{i=1}^{n-1}\sum_{j=1}^{n-1}\min(j,n-j,i)\right) \\ &= \frac{1}{12}(20n^3-30n^2+4n+3-3(-1)^n). \end{aligned}$$

Hence the combined total of the number of triangles and rectangles is

$$F_n = R_{n,1}+R_{n,2}+T_{n,1}+T_{n,2} = \frac{1}{12}(11n^4-2n^3-5n^2-22n+15-3(-1)^n).$$

The first eight numbers of this sequence are:

$$F_1 = 0,\ F_2 = 9,\ F_3 = 62,\ F_4 = 211,\ F_5 = 534,\ F_6 = 1127,\ F_7 = 2112,\ F_8 = 3629.$$

□