Problem 11045
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Prove that when n is a sufficiently large positive integer there exists a finite set S of prime numbers such that the sum of $\lfloor n/p \rfloor$ over $p \in S$ is equal to n.

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The proof is divided in two parts. In the first one, using Bertrand's Postulate, we show that for any integer $n \geq 1$ there exists a finite set T of primes with $|T| \leq \log_2 n + 1$ such that

$$n = \sum_{p \in T} \lfloor n/p \rfloor + |T|.$$

In the second part, using Prime Number Theorem, we prove that for n sufficiently large there exists a set T' of $|T|$ primes contained in $(n/2, n] \setminus T$.
Therefore

$$\sum_{p \in T'} \lfloor n/p \rfloor = \sum_{p \in T'} 1 = |T'| = |T|$$

and letting $S = T \cup T'$ we have that for n sufficiently large

$$\sum_{p \in S} \lfloor n/p \rfloor = \sum_{p \in T} \lfloor n/p \rfloor + \sum_{p \in T'} \lfloor n/p \rfloor = \sum_{p \in T} \lfloor n/p \rfloor + |T| = n$$

with $|S| \leq 2(\log_2 n + 1)$.

1) We first note that

for any integer $k \in [1, n]$ there is a prime p such that $\lfloor n/p \rfloor + 1 \in (k/2, k]$.

In fact, by Bertrand's Postulate, for any real number $x \geq 1$ there is a prime p such that $x < p \leq 2x$. Letting $x = n/k$ then

$$k/2 \leq n/p < k.$$

The first inequality implies that

$$k/2 \leq \lfloor 2n/p \rfloor / 2 \leq (2\lfloor n/p \rfloor + 1)/2 < \lfloor n/p \rfloor + 1,$$

whereas the second one gives

$$\lfloor n/p \rfloor + 1 \leq k.$$

Let p_j be the j-th prime number for $j \geq 1$. Now we prove by induction on m that for any integer $m \in [1, n]$ there exist $j_1 < j_2 < \cdots < j_N$ with $N \leq \log_2 m + 1$ such that

$$m = (\lfloor n/p_{j_1} \rfloor + 1) + (\lfloor n/p_{j_2} \rfloor + 1) + \cdots + (\lfloor n/p_{j_N} \rfloor + 1).$$

If $m = 1$ then there is j_1 such that $\lfloor n/p_{j_1} \rfloor + 1 \in (m/2, m] = \{1\}$ that is

$$1 = \lfloor n/p_{j_1} \rfloor + 1.$$

Let $m \geq 2$ and assume that the statement is true for $1, 2, \ldots, m-1$ then there is j_1 such that $\lfloor n/p_{j_1} \rfloor + 1 \in (m/2, m]$. Hence

$$0 \leq m - (\lfloor n/p_{j_1} \rfloor + 1) < m/2 \leq m - 1$$

and, by induction hypothesis, there exist $j_2 < \cdots < j_N$ with

$$N - 1 \le \log_2\left(m - (\lfloor n/p_{j_1} \rfloor + 1)\right) + 1 < \log_2(m/2) + 1 = \log_2 m$$

such that

$$m - (\lfloor n/p_{j_1} \rfloor + 1) = (\lfloor n/p_{j_2} \rfloor + 1) + \cdots + (\lfloor n/p_{j_N} \rfloor + 1)\,.$$

Note that $j_1 < j_2$ because $\lfloor n/p_{j_1} \rfloor + 1 > m/2$ and therefore

$$(\lfloor n/p_{j_2} \rfloor + 1) \le m - (\lfloor n/p_{j_1} \rfloor + 1) < (\lfloor n/p_{j_1} \rfloor + 1)\,.$$

So we have established that for any integer $n \ge 1$ there exists a finite set T of primes such that

$$n = \sum_{p \in T} (\lfloor n/p \rfloor + 1) = \sum_{p \in T} \lfloor n/p \rfloor + |T|$$

with $|T| \le \log_2 n + 1$.

2) The Prime Number Theorem says that

$$\lim_{n \to \infty} \frac{|\{p \in (1, n] : p \text{ is prime}\}|}{n/\log n} = 1.$$

Since $(n/2, n] = (1, n] \setminus (1, n/2]$ then

$$\lim_{n \to \infty} \frac{|\{p \in (n/2, n] : p \text{ is prime}\}|}{n/\log n} = 1/2,$$

and therefore, for n sufficiently large,

$$|\{p \in (n/2, n] : p \text{ is prime}\}| \ge \frac{n}{3 \log n} \ge 2(\log_2 n + 1) \ge 2|T|.$$

So it is possible to find $|T|$ primes in the interval $(n/2, n]$ which are different from those ones in T. This is the required set T'. $\square$