Problem 11039
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Proposed by N. Rosemberg and R. Stong (USA).

Let $\Delta_k = \{(x_1, \ldots, x_k) : x_i \geq 0$ *and* $\sum_{i=1}^k x_i \leq 1\}$ *and define* x_{k+1} *on* Δ_k *by* $x_{k+1} = 1 - \sum_{i=1}^k x_i$. *Suppose that* $a_1, \ldots, a_{k+1}$ *are distinct real numbers and that* f *is a* k *times differentiable function on* $[\min(a_i), \max(a_i)]$. *Prove that*

$$\int_{\Delta_k} f^{(k)} \left(\sum_{i=1}^{k+1} a_i x_i \right) dx_1 \ldots dx_k = D_f / D,$$

where

$$D = \begin{vmatrix} 1 & 1 & \ldots & 1 \\ a_1 & a_2 & \ldots & a_{k+1} \\ \vdots & \vdots & \ddots & \vdots \\ a_1^{k-1} & a_2^{k-1} & \ldots & a_{k+1}^{k-1} \\ a_1^k & a_2^k & \ldots & a_{k+1}^k \end{vmatrix}$$

and where D_f *is the same as* D *but with the last row replaced by*

$$(f(a_1), f(a_2), \ldots, f(a_{k+1})) .$$

Solution proposed by Roberto Tauraso, Dipartimento di Matematica, Università di Roma "Tor Vergata", via della Ricerca Scientifica, 00133 Roma, Italy.

The *Vandermonde determinant* of the numbers $z_1, \ldots, z_k$ is defined by

$$V_k(z_1, \ldots, z_k) = \begin{vmatrix} 1 & 1 & \ldots & 1 \\ z_1 & z_2 & \ldots & z_k \\ \vdots & \vdots & \ddots & \vdots \\ z_1^{k-1} & z_2^{k-1} & \ldots & z_k^{k-1} \end{vmatrix} = \prod_{i>j} (z_i - z_j).$$

The formula we are going to prove can be written in a more convenient way by expanding the determinant D_f along the last row

$$\begin{aligned} I_k(a_1, \ldots, a_{k+1}) &= \frac{\sum_{i=1}^{k+1} (-1)^{k+1+i} f(a_i) \cdot V_k(a_1, \ldots, \widehat{a_i}, \ldots, a_{k+1})}{V_{k+1}(a_1, \ldots, a_{k+1})} \\ &= \sum_{i=1}^{k+1} \frac{f(a_i)}{\prod_{j=1, j \neq i}^{k+1} (a_i - a_j)}. \end{aligned}$$

We will show that it holds by induction with respect to $k \geq 1$. For $k = 1$

$$\begin{aligned} I_1(a_1, a_2) &= \int_{\Delta_1} f'(a_1 x_1 + a_2 x_2)\, dx_1 = \int_0^1 f'(a_2 + (a_1 - a_2) x_1)\, dx_1 \\ &= \left[\frac{f(a_2 + (a_1 - a_2) x_1)}{a_1 - a_2} \right]_0^1 = \frac{f(a_2) - f(a_1)}{a_2 - a_1}. \end{aligned}$$

For $k > 1$, let $y_i = x_i$ for $i = 1, \ldots, k-1$ and let $y_k = 1 - \sum_{i=1}^{k-1} y_i$. Therefore, integrating with respect to the variable x_k, we obtain

$$\begin{aligned} I_k(a_1,\ldots,a_{k+1}) &= \int_{\Delta_{k-1}} dy_1 \ldots dy_{k-1} \int_0^{y_k} f^{(k)} \left(\sum_{i=1}^{k-1} a_i y_i + a_{k+1} y_k + (a_k - a_{k+1}) x_k \right) dx_k \\ &= \int_{\Delta_{k-1}} dy_1 \ldots dy_{k-1} \left[\frac{f^{(k-1)} \left(\sum_{i=1}^{k-1} a_i y_i + a_{k+1} y_k + (a_k - a_{k+1}) x_k \right)}{a_k - a_{k+1}} \right]_0^{y_k} \\ &= \frac{I_{k-1}(a_1,\ldots,a_{k-1},a_{k+1}) - I_{k-1}(a_1,\ldots,a_{k-1},a_k)}{a_{k+1} - a_k}. \end{aligned}$$

Now our formula can be easily verified using the inductive hypothesis:

$$\begin{aligned} I_k(a_1,\ldots,a_{k+1}) &= \frac{1}{a_{k+1} - a_k} \cdot \left[\sum_{i=1, i\neq k}^{k+1} \frac{f(a_i)}{\prod_{j=1, j\neq i,k}^{k+1} (a_i - a_j)} - \sum_{i=1}^{k} \frac{f(a_i)}{\prod_{j=1, j\neq i}^{k} (a_i - a_j)} \right] \\ &= \frac{1}{a_{k+1} - a_k} \cdot \left[\sum_{i=1}^{k-1} \frac{f(a_i)}{\prod_{j=1, j\neq i}^{k-1} (a_i - a_j)} \left(\frac{1}{a_i - a_{k+1}} - \frac{1}{a_i - a_k} \right) \right] \\ &\quad + \frac{f(a_k)}{\prod_{j=1, j\neq k}^{k+1} (a_k - a_j)} + \frac{f(a_{k+1})}{\prod_{j=1}^{k} (a_{k+1} - a_j)} = \sum_{i=1}^{k+1} \frac{f(a_i)}{\prod_{j=1, j\neq i}^{k+1} (a_i - a_j)}. \end{aligned}$$