Problem 11039

Proposed by Noah Rosemberg, University of Southern California, Los Angeles, CA, and Richard Stong, Rice University, Houston, TX.

Let \( \Delta_k = \{(x_1, \ldots, x_k) : x_i \geq 0 \text{ and } \sum_{i=1}^k x_i \leq 1\} \) and define \( x_{k+1} \) on \( \Delta_k \) by \( x_{k+1} = 1 - \sum_{i=1}^k x_i \). Suppose that \( a_1, \ldots, a_{k+1} \) are distinct real numbers and that \( f \) is a \( k \) times differentiable function on \([\min(a_i), \max(a_i)]\). Prove that

\[
\int_{\Delta_k} f^{(k)} \left( \sum_{i=1}^{k+1} a_i x_i \right) \, dx_1 \ldots dx_k = D_f/D,
\]

where

\[
D = \begin{vmatrix}
1 & 1 & \cdots & 1 \\
a_1 & a_2 & \cdots & a_{k+1} \\
\vdots & \vdots & \ddots & \vdots \\
a_{k-1} & a_{k-1} & \cdots & a_{k+1} \\
a_1 & a_k & \cdots & a_{k+1}
\end{vmatrix}
\]

and where \( D_f \) is the same as \( D \) but with the last row replaced by \((f(a_1), \ldots, f(a_{k+1}))\).

Solution proposed by Roberto Tauraso, Dipartimento di Matematica, Università di Roma “Tor Vergata”, via della Ricerca Scientifica, 00133 Roma, Italy.

The Vandermonde determinant of the numbers \( z_1, \ldots, z_k \) is defined by

\[
V_k(z_1, \ldots, z_k) = \begin{vmatrix}
1 & 1 & \cdots & 1 \\
z_1 & z_2 & \cdots & z_k \\
\vdots & \vdots & \ddots & \vdots \\
z_{k-1} & z_{k-1} & \cdots & z_k \\
z_1 & z_k & \cdots & z_k
\end{vmatrix} = \prod_{i>j} (z_i - z_j).
\]

The formula we are going to prove can be written in a more convenient way by expanding the determinant \( D_f \) along the last row

\[
I_k(a_1, \ldots, a_{k+1}) = \sum_{i=1}^{k+1} (-1)^{k+1+i} f(a_i) \cdot V_k(a_1, \ldots, \hat{a_i}, \ldots, a_{k+1})
\]

\[
= \sum_{i=1}^{k+1} f(a_i) \prod_{j=1, j\neq i}^{k+1} (a_i - a_j).
\]

We will show that it holds by induction with respect to \( k \geq 1 \). For \( k = 1 \)

\[
I_1(a_1, a_2) = \int_{\Delta_1} f'(a_1 x_1 + a_2 x_2) \, dx_1 = \int_0^1 f'(a_2 + (a_1 - a_2) x_1) \, dx_1 = \left[ \frac{f(a_2 + (a_1 - a_2) x_1)}{a_1 - a_2} \right]_0^1 = \frac{f(a_2) - f(a_1)}{a_2 - a_1}.
\]
For $k > 1$, let $y_i = x_i$ for $i = 1, \ldots, k - 1$ and let $y_k = 1 - \sum_{i=1}^{k-1} y_i$. Therefore, integrating with respect to the variable $x_k$, we obtain

$$I_k(a_1, \ldots, a_{k+1}) = \int_{\Delta_{k-1}} dy_1 \ldots dy_{k-1} \int_0^{y_k} f^{(k)} \left( \sum_{i=1}^{k-1} a_i y_i + a_{k+1} y_k + (a_k - a_{k+1}) x_k \right) dx_k$$

$$= \int_{\Delta_{k-1}} dy_1 \ldots dy_{k-1} \left[ \frac{f^{(k-1)} \left( \sum_{i=1}^{k-1} a_i y_i + a_{k+1} y_k + (a_k - a_{k+1}) x_k \right)}{a_k - a_{k+1}} \right]^{y_k}_0$$

$$= I_{k-1}(a_1, \ldots, a_{k-1}, a_{k+1}) - I_{k-1}(a_1, \ldots, a_{k-1}, a_k) \frac{a_{k+1} - a_k}{a_{k+1} - a_k}.$$

Now our formula can be easily verified using the inductive hypothesis:

$$I_k(a_1, \ldots, a_{k+1}) = \frac{1}{a_{k+1} - a_k} \left[ \sum_{i=1, i \neq k}^{k+1} f(a_i) \prod_{j=1, j \neq i, k}^{k+1} (a_i - a_j) - \sum_{i=1}^{k} \prod_{j=1, j \neq i}^{k} (a_i - a_j) \right]$$

$$= \frac{1}{a_{k+1} - a_k} \left[ \sum_{i=1}^{k-1} f(a_i) \prod_{j=1, j \neq i, k}^{k-1} (a_i - a_j) \left( \frac{1}{a_i - a_{k+1}} - \frac{1}{a_i - a_k} \right) \right]$$

$$+ \frac{f(a_{k+1})}{\prod_{j=1, j \neq k}^{k+1} (a_k - a_j)} + \frac{f(a_k)}{\prod_{j=1, j \neq k}^{k+1} (a_k - a_j)} = \sum_{i=1}^{k+1} f(a_i) \prod_{j=1, j \neq i}^{k+1} (a_i - a_j).$$