Problem 11037
(American Mathematical Monthly, Vol.110, October 2003)

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Let $\mathcal{B}$ and $\mathcal{C}$ be complex Banach spaces, and let $F : \mathcal{B} \to \mathcal{C}$ be a Fréchet differentiable map satisfying $\|F(x)\| \leq \|x\|$ for all $x \in \mathcal{B}$. Show that F is linear.

Solution proposed by Roberto Tauraso, Dipartimento di Matematica, Università di Roma "Tor Vergata", via della Ricerca Scientifica, 00133 Roma, Italy.

Since F is a Fréchet differentiable map then for all $x_0 \in \mathcal{B}$ there exists a bounded linear mapping $A_{x_0} : \mathcal{B} \to \mathcal{C}$ such that

$$\lim_{x \to x_0} \frac{\|F(x) - F(x_0) - A_{x_0}(x - x_0)\|}{\|x - x_0\|} = 0.$$

This is actually a complex differential and it is well known that F behaves like a holomorphic function: for all $\Psi \in \mathcal{C}'$ and for all $x \in \mathcal{B}$, the map

$$f(\zeta) = \Psi(F(\zeta x))$$

is holomorphic in $\mathbb{C}$. By the inequality $\|F(x)\| \leq \|x\|$, we have that $F(0) = 0$ and $f(0) = 0$. Hence also the map

$$g(\zeta) = \frac{f(\zeta)}{\zeta} = \Psi\left(\frac{F(\zeta x)}{\zeta}\right)$$

is holomorphic in $\mathbb{C}$ and

$$|g(\zeta)| \leq \|\Psi\| \cdot \frac{\|F(\zeta x)\|}{|\zeta|} \leq \|\Psi\| \cdot \frac{\|\zeta x\|}{|\zeta|} = \|\Psi\| \cdot \|x\|,$$

that is g is an entire bounded function and, by Liouville theorem, it is constant. Since $\mathcal{C}'$ separates points in $\mathcal{C}$ then also the map $F(\zeta x)/\zeta$ is equal to a constant (which depends on x): $F(\zeta x) = \zeta c_x$. The differentiability in $x_0 = 0$ yields

$$\lim_{\zeta \to 0} \frac{\|F(\zeta x) - F(0) - A_0(\zeta x)\|}{\|\zeta x\|} = \lim_{\zeta \to 0} \frac{\|\zeta c_x - \zeta A_0(x)\|}{\|\zeta x\|} = \frac{\|c_x - A_0 x\|}{\|x\|} = 0.$$

Hence $c_x = A_0 x$ and F is linear: $F(x) = A_0 x$. $\square$