Problem 11026
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Proposed by J. Sondow (USA).

Let H_n denote the nth harmonic number $\sum_1^n 1/k$. Let $H_0 = 0$. Prove that for positive integers n and k with $k \leq n$,

$$\sum_{i=0}^{k-1}\sum_{j=k}^{n}(-1)^{i+j-1}\binom{n}{i}\binom{n}{j}\frac{1}{j-i} = \sum_{i=0}^{k-1}\binom{n}{i}^2 (H_{n-i} - H_i).$$

Solution proposed by Roberto Tauraso, Dipartimento di Matematica, Università di Roma "Tor Vergata", via della Ricerca Scientifica, 00133 Roma, Italy.

We define for $i \neq j$

$$a_{ij} = (-1)^{i+j-1}\binom{n}{i}\binom{n}{j}\frac{1}{j-i} \quad \text{and} \quad b_i = \binom{n}{i}^2 (H_{n-i} - H_i).$$

Since $a_{ij} = -a_{ji}$ then

$$\sum_{i=0}^{k-1}\sum_{\substack{j=0\\ j\neq i}}^{k-1} a_{ij} = 0.$$

So we have to prove that

$$\sum_{i=0}^{k-1}\sum_{j=k}^{n} a_{ij} = \sum_{i=0}^{k-1}\sum_{j=k}^{n} a_{ij} + \sum_{i=0}^{k-1}\sum_{\substack{j=0\\ j\neq i}}^{k-1} a_{ij} = \sum_{i=0}^{k-1}\sum_{\substack{j=0\\ j\neq i}}^{n} a_{ij} = \sum_{i=0}^{k-1} b_i.$$

Therefore it suffices to show that for $i = 0, \ldots, n$

$$\sum_{\substack{j=0\\ j\neq i}}^{n} a_{ij} = b_i,$$

that is

$$\sum_{\substack{j=0\\ j\neq i}} \binom{n}{j}\frac{(-1)^j}{j-i} = (-1)^{i-1}\binom{n}{i}(H_{n-i} - H_i).$$

It is well known (see for example the beautiful book *Concrete Mathematics* of Graham-Knuth-Patashnik) that if f is a function then

$$\Delta^n f(x) = \sum_{j=0}^{n}\binom{n}{j}(-1)^j f(x+j).$$

In particular, when $f(x) = 1/x$, we obtain

$$\sum_{j=0}^{n}\binom{n}{j}\frac{(-1)^j}{x+j} = \frac{n!}{x(x+1)\cdots(x+n)} \quad \text{for } x \notin \{0, -1, \ldots, -n\}.$$

Hence, letting

$$P_i(x) = \prod_{j=0}^{i-1}(x+j) \quad \text{and} \quad Q_i(x) = \prod_{j=i+1}^{n}(x+j)$$

(note that $P_0(x) = Q_n(x) = 1$), we have

$$\sum_{\substack{j=0\\ j\neq i}}^{n} \binom{n}{j} \frac{(-1)^j}{x+j} = \frac{n!}{(x+i)\,P_i(x)\,Q_i(x)} + (-1)^{i-1}\binom{n}{i}\frac{1}{x+i}$$
$$= \frac{n! + (-1)^{i-1}\binom{n}{i} P_i(x)\,Q_i(x)}{(x+i)\,P_i(x)\,Q_i(x)}.$$

Now, the result follows by taking in the previous equation the limit as x goes to $-i$. The limit of the left side is just

$$\sum_{\substack{j=0\\ j\neq i}}^{n} \binom{n}{j} \frac{(-1)^j}{j-i}.$$

On the other hand, applying Hôpital theorem, the limit of the right side is

$$\lim_{x\to -i} \frac{n! + (-1)^{i-1}\binom{n}{i} P_i(x)\,Q_i(x)}{(x+i)\,P_i(x)\,Q_i(x)} \overset{\text{H}}{=} \lim_{x\to -i} \frac{(-1)^{i-1}\binom{n}{i}\,D(P_i(x)\,Q_i(x))}{P_i(x)\,Q_i(x) + (x+i)\,D(P_i(x)\,Q_i(x))}$$
$$= (-1)^{i-1}\binom{n}{i}\left(\frac{P_i'(-i)}{P_i(-i)} + \frac{Q_i'(-i)}{Q_i(-i)}\right)$$
$$= (-1)^{i-1}\binom{n}{i}(H_{n-i} - H_i),$$

because

$$\frac{P_i'(x)}{P_i(x)} = \sum_{j=0}^{i-1} \frac{1}{x+j} \quad \text{and} \quad \frac{Q_i'(x)}{Q_i(x)} = \sum_{j=i+1}^{n} \frac{1}{x+j}$$

and

$$\frac{P_i'(-i)}{P_i(-i)} = -H_i \quad \text{and} \quad \frac{Q_i'(-i)}{Q_i(-i)} = H_{n-i}.$$

□