Problem 11026

Proposed by Jonathan Sondow, New York, NY.

Let $H_n$ denote the $n$th harmonic number $\sum_1^n 1/k$. Let $H_0 = 0$. Prove that for positive integers $n$ and $k$ with $k \leq n$,

$$\sum_{i=0}^{k-1} \sum_{j=k}^{n} (-1)^{i+j-1} \binom{n}{i} \binom{n}{j} \frac{1}{j-i} = \sum_{i=0}^{k-1} \binom{n}{i} (H_{n-i} - H_i).$$

Solution proposed by Roberto Tauraso, Dipartimento di Matematica, Università di Roma “Tor Vergata”, via della Ricerca Scientifica, 00133 Roma, Italy.

We define for $i \neq j$

$$a_{ij} = (-1)^{i+j-1} \binom{n}{i} \binom{n}{j} \frac{1}{j-i} \quad \text{and} \quad b_i = \binom{n}{i}^2 (H_{n-i} - H_i).$$

Since $a_{ij} = -a_{ji}$ then

$$\sum_{i=0}^{k-1} \sum_{j=0}^{k-1} a_{ij} = 0.$$

So we have to prove that

$$\sum_{i=0}^{k-1} \sum_{j=k}^{n} a_{ij} = \sum_{i=0}^{k-1} \sum_{j=0}^{k-1} a_{ij} + \sum_{i=0}^{k-1} \sum_{j=0}^{k-1} a_{ij} = \sum_{i=0}^{k-1} \sum_{j=0}^{n} a_{ij} = \sum_{i=0}^{k-1} b_i.$$

Therefore it suffices to show that for $i = 0, \ldots, n$

$$\sum_{j=0}^{n} a_{ij} = b_i,$$

that is

$$\sum_{j=0}^{n} \binom{n}{j} \frac{(-1)^j}{j-i} = (-1)^{i-1} \binom{n}{i} (H_{n-i} - H_i).$$

It is well known (see for example the beautiful book Concrete Mathematics of Graham-Knuth-Patashnik) that if $f$ is a function then

$$\Delta^n f(x) = \sum_{j=0}^{n} \binom{n}{j} (-1)^j f(x+j).$$
In particular, when \( f(x) = 1/x \), we obtain
\[
\sum_{j=0}^{n} \binom{n}{j} \frac{(-1)^j}{x + j} = \frac{n!}{x(x+1) \cdots (x+n)} \quad \text{for } x \not\in \{0, -1, \ldots, -n\}.
\]

Hence, letting
\[
P_i(x) = \prod_{j=0}^{i-1} (x+j) \quad \text{and} \quad Q_i(x) = \prod_{j=i+1}^{n} (x+j)
\]
(note that \( P_0(x) = Q_n(x) = 1 \)), we have
\[
\sum_{j=0}^{n} \binom{n}{j} \frac{(-1)^j}{j - i} = \frac{n!}{(x+i) P_i(x) Q_i(x)} + (-1)^{i-1} \binom{n}{i} \frac{1}{x+i}
\]
\[
= \frac{n! + (-1)^{i-1} \binom{n}{i} P_i(x) Q_i(x)}{(x+i) P_i(x) Q_i(x)}.
\]

Now, the result follows by taking in the previous equation the limit as \( x \) goes to \(-i\). The limit of the left side is just
\[
\sum_{j=0}^{n} \binom{n}{j} \frac{(-1)^j}{j - i}.
\]

On the other hand, applying Hôpital theorem, the limit of the right side is
\[
\lim_{x \to -i} \frac{n! + (-1)^{i-1} \binom{n}{i} P_i(x) Q_i(x)}{(x+i) P_i(x) Q_i(x)} = \lim_{x \to -i} \frac{(-1)^{i-1} \binom{n}{i} \frac{D(P_i(x) Q_i(x))}{P_i(x) Q_i(x) + (x+i) D(P_i(x) Q_i(x))}}{P_i(x) Q_i(x) + (x+i) D(P_i(x) Q_i(x))}
\]
\[
= (-1)^{i-1} \binom{n}{i} \left( \frac{P_i'(-i)}{P_i(-i)} + \frac{Q_i'(-i)}{Q_i(-i)} \right)
\]
\[
= (-1)^{i-1} \binom{n}{i} (H_{n-i} - H_i),
\]

because
\[
\frac{P_i'(x)}{P_i(x)} = \sum_{j=0}^{i-1} \frac{1}{x+j} \quad \text{and} \quad \frac{Q_i'(x)}{Q_i(x)} = \sum_{j=i+1}^{n} \frac{1}{x+j}
\]
and
\[
\frac{P_i'(-i)}{P_i(-i)} = -H_i \quad \text{and} \quad \frac{Q_i'(-i)}{Q_i(-i)} = H_{n-i}.
\]