Problem 11023
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Proposed by W. W. Chao (China).

Find all pairs (x, y) of integers such that

$$x^2 + 3xy + 4006(x + y) + 2003^2 = 0.$$

Solution proposed by Roberto Tauraso, Dipartimento di Matematica, Università di Roma "Tor Vergata", via della Ricerca Scientifica, 00133 Roma, Italy.

The given equation represents a hyperbola and, after multiplying by 9, it can be factorized in this way

$$(3x + 2p) \cdot (3x + 9y + 4p) = -p^2$$

where $p = 2003$. The integer factors $u = 3x + 2p$ and $v = 3x + 9y + 4p$ divide $-p^2$ and since p is prime they can assume only the following values: ± 1, $\pm p$ and $\pm p^2$. Moreover, solving with respect to x and y, we have that

$$x = \frac{u - 2p}{3}\ , \quad y = \frac{v - u - 2p}{9}.$$

Hence we need only consider the following cases:

u	-1	$-p$	$-p^2$	1	p	p^2
v	p^2	p	1	$-p^2$	$-p$	-1
x	$-\frac{2p+1}{3}$	$-p$	$-\frac{p(p+2)}{3}$	$-\frac{2p-1}{3}$	$-\frac{p}{3}$	$\frac{p(p-2)}{3}$
y	$\left(\frac{p-1}{3}\right)^2$	0	$\left(\frac{p-1}{3}\right)^2$	$-\left(\frac{p+1}{3}\right)^2$	$-\frac{4p}{9}$	$-\left(\frac{p+1}{3}\right)^2$

Since 2003 is equal to 2 modulo 3, the integer solutions (x, y) are precisely:

$$\begin{aligned}
(-p, 0) &= (-2003, 0),\\
\left(-\frac{2p-1}{3}, -\left(\frac{p+1}{3}\right)^2\right) &= (-1335, -446224),\\
\left(\frac{p(p-2)}{3}, -\left(\frac{p+1}{3}\right)^2\right) &= (1336001, -446224).
\end{aligned}$$

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