Problem 11014
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Let $f : \mathbb{R} \to \mathbb{R}$ be differentiable, with $f(0) = 1$, and with f and f' both nowhere zero on $\mathbb{R}$. Let a_1 be a positive real number, and for $n \geq 1$ let $a_{n+1} = a_n\, f(a_n)$. Prove that $\sum_{n=1}^{\infty} a_n$ is divergent.

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The function f is injective otherwise there are two points $x_1 < x_2$ such that $f(x_1) = f(x_2)$ and, by Rolle's theorem, there would be a point inside the interval (x_1, x_2) where f' vanishes. Since f is continuous this means that f is positive and strictly monotone.
However the statement holds in a more general setting: it suffices that f is positive and

$$\liminf_{x \to 0^+} \frac{f(x) - 1}{x} \neq -\infty.$$

Note that $f(x) = 1/(1 + x^{1/t})^t$ generates the sequence $a_n = 1/n^t$ and the above limit is equal to $-\infty$ iff $t > 1$ that is when the series is convergent.
Assume that the positive sequence $\{a_n\}$ goes to zero (otherwise the series is trivially divergent). Then there are $\alpha, \delta > 0$ such that $f(x) \geq 1 - \alpha x > 0$ for $0 < x < \delta$ and

$$\frac{1}{x\, f(x)} < \frac{1}{x} \cdot \frac{1}{1 - \alpha x} \leq \frac{1}{x} \cdot (1 + \alpha\, x + \mathrm{o}(x)) \leq \frac{1}{x} + \alpha + \mathrm{o}(1) < \frac{1}{x} + 2\,\alpha.$$

Now, let $n_0 \geq 1$ such that $0 < a_n < \delta$ for all $n > n_0$ then

$$\frac{1}{a_n} = \frac{1}{a_{n-1}\, f(a_{n-1})} < \frac{1}{a_{n-1}} + 2\,\alpha < \frac{1}{a_{n-2}} + 4\,\alpha \; < \cdots < \frac{1}{a_{n_0}} + 2\,(n - n_0)\,\alpha.$$

Thus the series is divergent because for $n > n_0$

$$a_n > \frac{1}{\dfrac{1}{a_{n_0}} + 2\,(n - n_0)\,\alpha} \sim \frac{1}{n}.$$

$\square$