Problem 11013
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Proposed by D. Callan (USA).

A Dyck n-path is a lattice path of n upsteps $(1,1)$ and n downsteps $(1,-1)$ that starts at the origin and never falls below the x-axis. Show that the number of Dyck $(2n)$-paths that avoid $\{(4k,0) : 1 \leq k \leq n-1\}$ is twice the number of Dyck $(2n-1)$-paths.

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It is well known that the total number of Dyck n-paths is equal to the catalan number

$$C_n = \frac{1}{n+1}\binom{2n}{n}.$$

Let A_n be the number of Dyck $(2n)$-paths that avoid the set

$$P_n = \{(4k,0) : 1 \leq k \leq n-1\},$$

then we have to prove that

$$A_n = 2C_{2n-1} \quad \text{for all } n \geq 1.$$

By the inclusion-exclusion principle,

$$A_n = \sum_{j=0}^{\infty}(-1)^j \cdot N_{j,n}$$

where $N_{j,n}$ is the number of Dyck n-paths which hit the set P_n at least $j \geq 0$ times. Note that this sum is finite actually, because $N_{j,n} = 0$ for $j \geq n$.
Since a Dyck n-paths which hit the points $(4k_1,0), \cdots, (4k_j,0) \in P_n$ with $0 = k_0 < k_1 < \cdots < k_j < k_{j+1} = n$ can be obtained by concatening $j+1$ Dyck paths respectively of lengths $4s_i = 4(k_i - k_{i-1})$ for $i = 1, \cdots, j+1$, then the total number of such paths is

$$\prod_{i=1}^{j+1} C_{2s_i}.$$

Varying the j points in P_n, we get the following formula for $N_{j,n}$

$$N_{j,n} = \sum_{\substack{s_1+\cdots+s_{j+1}=n \\ s_1\geq 1,\cdots,s_{j+1}\geq 1}} \left(\prod_{i=1}^{j+1} C_{2s_i}\right).$$

Let $C(x)$ be the generating function of the catalan numbers and define the functions

$$G(x) = \frac{1}{2}\left(C(x)+C(-x)\right) - 1 = \sum_{n=1}^{\infty} C_{2n}x^{2n},$$

$$F(x) = \frac{x}{2}\left(C(x)-C(-x)\right) = \sum_{n=1}^{\infty} C_{2n-1}x^{2n}.$$

By the convolution property and the formula for $N_{j,n}$, we have that

$$\sum_{n=1}^{\infty} N_{j,n}x^{2n} = G^{j+1}(x)$$

and

$$\begin{aligned}\sum_{n=1}^{\infty} A_n x^{2n} &= \sum_{n=1}^{\infty}\left(\sum_{j=0}^{\infty}(-1)^j \cdot N_{j,n}\right) x^{2n} \\ &= \sum_{j=0}^{\infty}(-1)^j \cdot \left(\sum_{n=1}^{\infty} N_{j,n} x^{2n}\right) \\ &= \sum_{j=0}^{\infty}(-1)^j \cdot G^{j+1}(x) \\ &= \frac{G(x)}{1+G(x)}.\end{aligned}$$

Therefore, in order to prove $A_n = 2C_{2n-1}$ for all $n \geq 1$, it suffices to show that

$$\sum_{n=1}^{\infty} A_n x^{2n} = \frac{G(x)}{1+G(x)} = 2F(x) = \sum_{n=1}^{\infty} 2C_{2n-1} x^{2n}.$$

This is equivalent to

$$4F(x)\left(1+G(x)\right) - 2G(x) = \left[xC^2(x) - C(x) + 1\right] + \left[(-x)C^2(-x) - C(-x) + 1\right] = 0,$$

which certainly holds because the generating function $C(x)$ satisfies the equation

$$xC^2 - C(x) + 1 = 0.$$

□