Problem 11012
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Given a positive integer n, find the minimum value of

$$\frac{x_1^3 + \cdots + x_n^3}{x_1 + \cdots + x_n}$$

subject to the condition that $x_1, \ldots, x_n$ be distinct positive integers.

Solution proposed by Roberto Tauraso, Dipartimento di Matematica, Università di Roma "Tor Vergata", via della Ricerca Scientifica, 00133 Roma, Italy.

We will prove that the minimum value of the function

$$F(x_1, x_2, \cdots, x_n) = \frac{x_1^3 + \cdots + x_n^3}{x_1 + \cdots + x_n}$$

in the set $A_n = \{0 < x_1 < x_2 < \cdots < x_n \,:\, x_k \in \mathbb{N}\}$ is

$$F(1, 2, \cdots, n) = \frac{1^3 + 2^3 + \cdots + n^3}{1 + 2 + \cdots + n} = \frac{\left(\frac{n(n+1)}{2}\right)^2}{\frac{n(n+1)}{2}} = \frac{n(n+1)}{2}.$$

The statement is trivial for $n = 1$, so assume that $n \geq 2$. Note that the minimum value has to be attained in the finite set $A_n \cap \{|x_n| \leq n^2\}$ because for $x_n > n^2$

$$\frac{x_1^3 + \cdots + x_n^3}{x_1 + \cdots + x_n} > \frac{x_n^3}{n \cdot x_n} = \frac{x_n^2}{n} > n^3 > \frac{n(n+1)}{2}.$$

Let $(x_1, x_2, \ldots, x_n)$ be a minimum point then it suffices to show that

$$x_n - x_{n-1} = x_{n-1} - x_{n-2} = \cdots = x_2 - x_1 = x_1 - x_0 = 1 \qquad (x_0 = 0).$$

If these conditions are not satisfied then there is an integer $1 \leq r \leq n$ such that the numbers $x_{n-r+1}, \cdots, x_n$ are consecutive whereas $x_{n-r+1} - x_{n-r} > 1$. This gives us a contradiction: the value can be lowered by extending the sequence of consecutive numbers, that is by replacing $x_{n-r+1} + k$ with $x_{n-r} + k + 1$ (smaller) in $x_{n-r+k+1}$ for $k = 0, \ldots, r-1$

$$F(x_1, \cdots, x_{n-r+1}, \cdots, x_{n-r+1} + r - 1) > F(x_1, \cdots, x_{n-r} + 1, \cdots, x_{n-r} + r).$$

The above inequality holds because the function

$$f_r(x) = \frac{\displaystyle\sum_{k=1}^{n-r} x_k^3 + \sum_{k=1}^{r} (x+k)^3}{\displaystyle\sum_{k=1}^{n-r} x_k + \sum_{k=1}^{r} (x+k)}$$

is strictly increasing for $x \geq x_{n-r}$: the derivative is positive if

$$3 \cdot \sum_{k=1}^{r} (x+k)^2 \cdot \left(\sum_{k=1}^{n-r} x_k + \sum_{k=1}^{r} (x+k) \right) > r \left(\sum_{k=1}^{n-r} x_k^3 + \sum_{k=1}^{r} (x+k)^3 \right).$$

Since

$$3 \cdot \sum_{k=1}^{r} (x+k)^2 \cdot \sum_{k=1}^{n-r} x_k \geq 3 \cdot r \cdot \sum_{k=1}^{n-r} x^2 \cdot x_k \geq r \cdot \sum_{k=1}^{n-r} x_k^3,$$

it remains to prove that

$$3 \cdot \sum_{k=1}^{r} (x+k)^2 \cdot \sum_{k=1}^{r} (x+k) > r \cdot \sum_{k=1}^{r} (x+k)^3.$$

This follows by noting that the function

$$g_r(x) = 3 \cdot \sum_{k=1}^{r} (x+k)^2 \cdot \sum_{k=1}^{r} (x+k) - r \cdot \sum_{k=1}^{r} (x+k)^3$$

is positive for $x \geq 0$. In fact

$$\begin{aligned} g_r(0) &= 3 \cdot \sum_{k=1}^{r} k^2 \cdot \sum_{k=1}^{r} k - r \sum_{k=1}^{r} k^3 \\ &= 3 \cdot \frac{r(r+1)(2r+1)}{6} \cdot \frac{r(r+1)}{2} - r \cdot \left(\frac{r(r+1)}{2} \right)^2 \\ &= (r+1) \cdot \left(\frac{r(r+1)}{2} \right)^2 > 0 \end{aligned}$$

and its derivative is positive for $x \geq 0$

$$\begin{aligned} g_r'(x) &= 6 \cdot \sum_{k=1}^{r} (x+k) \cdot \sum_{k=1}^{r} (x+k) + 3 \cdot r \cdot \sum_{k=1}^{r} (x+k)^2 - 3 \cdot r \cdot \sum_{k=1}^{r} (x+k)^2 \\ &= 6 \cdot \left(\sum_{k=1}^{r} (x+k) \right)^2 > 0. \end{aligned}$$

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