Problem 11008
(American Mathematical Monthly, Vol.110, April 2003)

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Let $A(z) = \sum_{k=0}^{n} a_k z^k$ *be a monic polynomial with complex coefficients and with zeros* $z_1, \ldots, z_n$. *Prove that*

$$\frac{1}{n}\sum_{k=1}^{n} |z_k|^2 < 1 + \max_{1\le k\le n} |a_{n-k}|^2.$$

Solution proposed by Roberto Tauraso, Dipartimento di Matematica, Università di Roma "Tor Vergata", via della Ricerca Scientifica, 00133 Roma, Italy.

We will prove that

$$\sum_{k=1}^{n} |z_k|^2 \le n - 1 + \sum_{k=1}^{n} |a_{n-k}|^2,$$

then the required inequality follows easily:

$$\frac{1}{n}\sum_{k=1}^{n} |z_k|^2 \le 1 - \frac{1}{n} + \frac{1}{n}\sum_{k=1}^{n} |a_{n-k}|^2 < 1 + \frac{1}{n}\sum_{k=1}^{n} |a_{n-k}|^2 \le 1 + \max_{1\le k\le n} |a_{n-k}|^2.$$

Let M be the $n \times n$ companion matrix for the monic polynomial $A(z)$

$$M = \begin{pmatrix} 0 & 0 & \cdots & 0 & -a_0 \\ 1 & 0 & \cdots & 0 & -a_1 \\ 0 & 1 & \cdots & 0 & -a_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & -a_{n-1} \end{pmatrix}.$$

By Schur's theorem, there exist an upper triangular matrix T and a unitary matrix U such that $M = U\,T\,U^*$. Recall that the Frobenius norm of a $n \times n$ complex matrix $C = [c_{ij}]$ is defined as $\|C\|_F = \sqrt{\sum_{i,j=1}^{n} |c_{ij}|^2})$. Since this norm is invariant with respect to the multiplication by a unitary matrix we have that

$$\|M\|_F = \|U\,T\,U^*\|_F = \|U\,T\|_F = \|T\|_F.$$

Note that the main diagonal of T consists of the eigenvalues of M which are the zeroes $z_1, \ldots, z_n$ of its characteristic polynomial $A(z)$. Therefore, computing the norms,

$$\sum_{k=1}^{n} |z_k|^2 \le \|T\|_F^2 = \|M\|_F^2 = n - 1 + \sum_{k=1}^{n} |a_{n-k}|^2.$$

□