Problem 11007
(American Mathematical Monthly, Vol.110, April 2003)

Proposed by Western Maryland College Problems Group (USA).

Let $\left\langle \; \right\rangle$ *denote Eulerian numbers, and let* $\left\{ \; \right\}$ *denote Stirling numbers of the second kind. Show that*

$$\sum_{j=1}^{n} 2^{j-1} \left\langle {n \atop j} \right\rangle = \sum_{j=1}^{n} j\,! \left\{ {n \atop j} \right\}.$$

Solution proposed by Roberto Tauraso, Dipartimento di Matematica, Università di Roma "Tor Vergata", via della Ricerca Scientifica, 00133 Roma, Italy.

We will prove these two equalities by induction on $n \geq 1$:

$$\sum_{j=1}^{n} \left\langle {n \atop j} \right\rangle \frac{x^{n+1-j}}{(1-x)^{n+1}} = \sum_{j=1}^{\infty} j^n\, x^j = \sum_{j=1}^{n} j\,! \left\{ {n \atop j} \right\} \frac{x^j}{(1-x)^{j+1}} \quad \text{for } |x| < 1.$$

The wanted identity follows taking $x = \frac{1}{2}$.
We start with the first one. Since $\left\langle {1 \atop 1} \right\rangle = 1$ and $\frac{x}{(1-x)^2} = \sum_{j=1}^{\infty} j^n\, x^j$ then the equality holds for $n = 1$. Moreover, for $n \geq 1$ we have that

$$\begin{aligned}
\sum_{j=1}^{\infty} j^{n+1}\, x^j &= x \cdot \frac{d}{dx}\left(\sum_{j=1}^{\infty} j^n\, x^j\right) = x \cdot \frac{d}{dx}\left(\sum_{j=1}^{n} \left\langle {n \atop j} \right\rangle \frac{x^{n+1-j}}{(1-x)^{n+1}}\right) \\
&= x \cdot \sum_{j=1}^{n} \left\langle {n \atop j} \right\rangle \left((n+1-j)\, \frac{x^{n-j}}{(1-x)^{n+1}} + (n+1)\, \frac{x^{n+1-j}}{(1-x)^{n+2}} \right) \\
&= \sum_{j=1}^{n} \left\langle {n \atop j} \right\rangle \left((n+1-j)\, \frac{x^{n+1-j}(1-x)}{(1-x)^{n+2}} + (n+1)\, \frac{x^{n+2-j}}{(1-x)^{n+2}} \right) \\
&= \sum_{j=1}^{n} \left\langle {n \atop j} \right\rangle \left((n+1-j)\, \frac{x^{n+1-j}}{(1-x)^{n+2}} + j\, \frac{x^{n+2-j}}{(1-x)^{n+2}} \right) \\
&= \sum_{j=2}^{n+1} (n+2-j) \left\langle {n \atop j-1} \right\rangle \frac{x^{n+2-j}}{(1-x)^{n+2}} + \sum_{j=1}^{n} j \left\langle {n \atop j} \right\rangle \frac{x^{n+2-j}}{(1-x)^{n+2}} \\
&= \sum_{j=1}^{n+1} \left\langle {n+1 \atop j} \right\rangle \frac{x^{n+2-j}}{(1-x)^{n+2}}.
\end{aligned}$$

The last equality holds because Eulerian numbers satisfy the following recurrence relation (see for example the *On-Line Encyclopedia of Integer Sequences*):

$$\left\langle {n \atop 1} \right\rangle = \left\langle {n \atop n} \right\rangle = 1 \quad \text{and} \quad \left\langle {n+1 \atop j} \right\rangle = (n+2-j) \left\langle {n \atop j-1} \right\rangle + j \left\langle {n \atop j} \right\rangle \text{ for } j = 2, \ldots, n.$$

Now the second one. Since $\left\{ {1 \atop 1} \right\} = 1$ and $\frac{x}{(1-x)^2} = \sum_{j=1}^{\infty} j^n\, x^j$ then the equality holds for $n = 1$.

Moreover, for $n \geq 1$ we have that

$$\begin{aligned}
\sum_{j=1}^{\infty} j^{n+1}\, x^j &= x \cdot \frac{d}{dx} \left(\sum_{j=1}^{\infty} j^n\, x^j \right) = x \cdot \frac{d}{dx} \left(\sum_{j=1}^{n} j\,! \left\{ {n \atop j} \right\} \frac{x^j}{(1-x)^{j+1}} \right) \\
&= x \cdot \sum_{j=1}^{n} \left\{ {n \atop j} \right\} \left(j\,!\, j\, \frac{x^{j-1}}{(1-x)^{j+1}} + j\,!\,(j+1)\, \frac{x^j}{(1-x)^{j+2}} \right) \\
&= \sum_{j=1}^{n} \left\{ {n \atop j} \right\} \left(j\,!\, j\, \frac{x^{j}}{(1-x)^{j+1}} + (j+1)\,!\, \frac{x^{j+1}}{(1-x)^{j+2}} \right) \\
&= \sum_{j=1}^{n} j\,!\, j \left\{ {n \atop j} \right\} \frac{x^j}{(1-x)^{j+1}} + \sum_{j=2}^{n+1} j\,! \left\{ {n \atop j-1} \right\} \frac{x^j}{(1-x)^{j+1}} \\
&= \sum_{j=1}^{n+1} j\,! \left\{ {n+1 \atop j} \right\} \frac{x^j}{(1-x)^{j+1}}.
\end{aligned}$$

The last equality holds because Stirling numbers of the second kind satisfy the following recurrence relation:

$$\left\{ {n \atop 1} \right\} = \left\{ {n \atop n} \right\} = 1 \quad \text{and} \quad \left\{ {n+1 \atop j} \right\} = j \left\{ {n \atop j} \right\} + \left\{ {n \atop j-1} \right\} \text{ for } j = 2, \ldots, n.$$

□