Problem 11002
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Pooh Bear has $2N+1$ honey pots. No matter which one of them he sets aside, he can split the remaining $2N$ pots into two sets of the same total weight, each consisting of N pots. Must all $2N+1$ pots weigh the same?

Solution proposed by Roberto Tauraso, Dipartimento di Matematica, Università di Roma "Tor Vergata", via della Ricerca Scientifica, 00133 Roma, Italy.

The answer is yes. Let $x = (x_1, x_2, \ldots, x_{2N+1})$ be the vector of the weights. We know that taking away any x_i it is possible to split the other $2N$ components into two sets of N elements each such that they have the same total sum. This property can be stated in the following way: there exists a $(2N+1)\times(2N+1)$ matrix A whose main diagonal is zero , in each row N coefficients are equal to 1 and the remaining N are equal to -1 and such that $A\,x = 0$.
In order to prove that the weights are all the same we have to show that

$$\mathrm{Ker}(A) = \mathrm{span}\{(1,\ldots,1)\}.$$

Of course $(1,\ldots,1) \in \mathrm{Ker}(A)$, hence it suffices to prove that

$$\dim(\mathrm{Ker}(A)) = 1 \quad \text{that is} \quad \mathrm{rank}(A) = 2N.$$

This is equivalent to show that $\det(B) \neq 0$ where B is the $2N\times 2N$ matrix obtained by deleting the last row and the last column of A. Actually we will prove that $\det(B) \neq 0 \mod 2$. This determinant is easier to compute because we do not need to know the sign of the non-zero elements of B. If we denote with M_n the $n\times n$ matrix which has all coefficients equal to 1 unless the elements of the main diagonal which are equal to 0 then $M_n = B \mod 2$ and

$$\det(M_n) = (-1)^{n-1}\cdot(n-1).$$

Therefore

$$\det(B) = \det(M_{2N}) = -(2N-1) = 1 \neq 0 \mod 2.$$

□

Remark: the formula $\det(M_n) = (-1)^{n-1}\cdot(n-1)$ can be easily proven by induction. For $n=1$ it is trivial. Now assume that $n \geq 1$ and let $\{e_1,\ldots,e_n\}$ be the natural n-base then

$$\begin{aligned}
\det(M_{n+1}) &= \det(1-e_1,\ldots,1-e_n,1-e_{n+1})\\
&= \det(1-e_1,\ldots,1-e_n,1) - \det(1-e_1,\ldots,1-e_n,e_{n+1})\\
&= (-1)^n\cdot\det(e_1,\ldots,e_n,1) - \det(M_n)\\
&= (-1)^n\cdot\det(e_1,\ldots,e_n,1-e_1-\cdots-e_n) - (-1)^{n-1}\cdot(n-1)\\
&= (-1)^n\cdot\det(e_1,\ldots,e_n,e_{n+1}) + (-1)^n\cdot(n-1)\\
&= (-1)^n + (-1)^n\cdot(n-1) = (-1)^n\cdot n.
\end{aligned}$$