Problem 11002

Proposed by Y. F. S. Pétermann (Switzerland).

Pooh Bear has $2N + 1$ honey pots. No matter which one of them he sets aside, he can split the remaining $2N$ pots into two sets of the same total weight, each consisting of $N$ pots. Must all $2N + 1$ pots weigh the same?

Solution proposed by Roberto Tauraso, Dipartimento di Matematica, Università di Roma “Tor Vergata”, via della Ricerca Scientifica, 00133 Roma, Italy.

The answer is yes. Let $x = (x_1, x_2, \ldots, x_{2N+1})$ be the vector of the weights. We know that taking away any $x_i$ it is possible to split the other $2N$ components into two sets of $N$ elements each such that they have the same total sum. This property can be stated in the following way: there exists a $(2N + 1) \times (2N + 1)$ matrix $A$ whose main diagonal is zero, in each row $N$ coefficients are equal to 1 and the remaining $N$ are equal to $-1$ and such that $Ax = 0$.

In order to prove that the weights are all the same we have to show that 

$$\text{Ker}(A) = \text{span}\{(1, \ldots, 1)\}.$$ 

Of course $(1, \ldots, 1) \in \text{Ker}(A)$, hence it suffices to prove that 

$$\text{dim}(\text{Ker}(A)) = 1 \quad \text{that is} \quad \text{rank}(A) = 2N.$$ 

This is equivalent to show that $\det(B) \neq 0$ where $B$ is the $2N \times 2N$ matrix obtained by deleting the last row and the last column of $A$. Actually we will prove that $\det(B) \neq 0 \mod 2$. This determinant is easier to compute because we do not need to know the sign of the non-zero elements of $B$. If we denote with $M_n$ the $n \times n$ matrix which has all coefficients equal to 1 unless the elements of the main diagonal which are equal to 0 then $M_n = B \mod 2$ and 

$$\det(M_n) = (-1)^{n-1} \cdot (n - 1).$$ 

Therefore 

$$\det(B) = \det(M_{2N}) = -(2N - 1) = 1 \neq 0 \mod 2.$$ 

□

Remark: the formula $\det(M_n) = (-1)^{n-1} \cdot (n - 1)$ can be easily proven by induction. For $n = 1$ it is trivial. Now assume that $n \geq 1$ and let $\{e_1, \ldots, e_n\}$ be the natural $n$-base then 

$$\det(M_{n+1}) = \det(1 - e_1, \ldots, 1 - e_n, 1 - e_{n+1})$$

$$\det(1 - e_1, \ldots, 1 - e_n, 1) - \det(1 - e_1, \ldots, 1 - e_n, e_{n+1})$$

$$= (-1)^n \cdot \det(e_1, \ldots, e_n, 1) - \det(M_n)$$

$$= (-1)^n \cdot \det(1, \ldots, 1 - e_1 - \cdots - e_n) - (-1)^{n-1} \cdot (n - 1)$$

$$= (-1)^n \cdot \det(e_1, \ldots, e_n, e_{n+1}) + (-1)^n \cdot (n - 1)$$

$$= (-1)^n + (-1)^n \cdot (n - 1) = (-1)^n \cdot n.$$