Problem 11002

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Pooh Bear has \(2N + 1\) honey pots. No matter which one of them he sets aside, he can split the remaining \(2N\) pots into two sets of the same total weight, each consisting of \(N\) pots. Must all \(2N + 1\) pots weigh the same?

Solution proposed by Roberto Tauraso, Dipartimento di Matematica, Università di Roma “Tor Vergata”, via della Ricerca Scientifica, 00133 Roma, Italy.

The answer is yes. Let \(x = (x_1, x_2, \ldots, x_{2N+1})\) be the vector of the weights. We know that taking away any \(x_i\) it is possible to split the other \(2N\) components into two sets of \(N\) elements each such that they have the same total sum. This property can be stated in the following way: there exists a \((2N+1) \times (2N+1)\) matrix \(A\) whose main diagonal is zero, in each row \(N\) coefficients are equal to 1 and the remaining \(N\) are equal to \(-1\) and such that \(A x = 0\).

In order to prove that the weights are all the same we have to show that \(\text{Ker}(A) = \text{span}\{(1, \ldots, 1)\}\).

Of course \((1, \ldots, 1) \in \text{Ker}(A)\), hence it suffices to prove that \(\dim(\text{Ker}(A)) = 1\) that is \(\text{rank}(A) = 2N\).

This is equivalent to show that \(\det(B) \neq 0\) where \(B\) is the \(2N \times 2N\) matrix obtained by deleting the last row and the last column of \(A\). Actually we will prove that \(\det(B) \neq 0 \ mod \ 2\). This determinant is easier to compute because we do not need to know the sign of the non-zero elements of \(B\). If we denote with \(M_n\) the \(n \times n\) matrix which has all coefficients equal to 1 unless the elements of the main diagonal are equal to 0 then \(M_n = B \ mod \ 2\) and \(\det(M_n) = (-1)^n \cdot (n-1)\).

Therefore \(\det(B) = \det(M_{2N}) = -(2N - 1) \neq 0 \ mod \ 2\).

Remark: the formula \(\det(M_n) = (-1)^{n-1} \cdot (n-1)\) can be easily proven by induction. For \(n = 1\) it is trivial. Now assume that \(n \geq 1\) and let \(\{e_1, \ldots, e_n\}\) be the natural \(n\)-base then

\[
\det(M_{n+1}) = \det(1 - e_1, \ldots, 1 - e_n, 1 - e_{n+1}) = \\
\det(1 - e_1, \ldots, 1 - e_{n}, 1) - \det(1 - e_1, \ldots, 1 - e_n, e_{n+1}) = \\
(-1)^n \cdot \det(e_1, \ldots, e_{n+1}) - \det(M_n) = \\
(-1)^n \cdot \det(e_1, \ldots, e_n, 1 - e_1 - \cdots - e_n) - (-1)^{n-1} \cdot (n-1) = \\
(-1)^n \cdot \det(e_1, \ldots, e_n, e_{n+1}) + (-1)^n \cdot (n-1) = \\
(-1)^n + (-1)^n \cdot (n-1) = (-1)^n \cdot n.
\]