Problem 10998

Proposed by R. Tauraso (Italy).

Let $D$ be a non-empty, open, connected, and relatively compact set in a metric space $X$ with metric $d$. Prove that if $f$ is a continuous map from $D$ into $D$ such that $f(D)$ is open, then there exists a point $x_0 \in D$ such that

$$d(x_0, \partial D) = d(f(x_0), \partial D).$$

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Assume that there is no such point $x_0$ then we will get a contradiction. The two sets

$$D^+ := \{ x \in D : d(f(x), \partial D) > d(x, \partial D) \}, \quad D^- := \{ x \in D : d(f(x), \partial D) < d(x, \partial D) \}$$

are disjoint, open ($f$ and $d(\cdot, \partial D)$ are continuous maps) and their union is equal to $D$. Since $D$ is connected, one of these two sets is empty.

The set $D$ is compact, hence there is a point $c \in D$ (it is not unique in general) such that

$$d(c, \partial D) = d_M := \max_{x \in \overline{D}} d(x, \partial D).$$

Moreover, $d_M > 0$ because $D$ is open and non-empty. Thus $c$ stays inside $D$ and since it reaches the maximal distance from the boundary, it belongs to $D^-$. Therefore the set $D^+$ has to be empty and $D = D^-$. This fact gives that for all $r > 0$, $f^{-1}(D_r) \subset D_r$, where $D_r$ is the compact set $\{ x \in D : d(x, \partial D) \geq r \}$.

We claim that the range set $f(D)$ is closed in $D$. Let $\{y_n\}_{n \geq 0}$ be a sequence in $f(D)$ which converges to $y \in D$ and let $r$ be the distance of the compact set $\{y_n\}_{n \geq 0} \cup \{y\}$ from the boundary $\partial D$. Therefore $\{y_n\}_{n \geq 0} \cup \{y\} \subset D_r$ and the inclusion $f^{-1}(D_r) \subset D_r$ implies that if $\{x_n\}_{n \geq 0}$ is a sequence in $D$ such that $f(x_n) = y_n$ then it is contained in $D_r$ and, since $D_r$ is compact, it converges, up to a subsequence, to some $x \in D$. Hence, by the continuity of $f$, $y = f(x) \in f(D)$.

So $f(D)$ is a non-empty open and closed subset of the connected set $D$ that is $f(D) = D$. In particular there exists $x \in D$ such that $f(x) = c$ and

$$d_M = d(c, \partial D) = d(f(x), \partial D) < d(x, \partial D).$$

against the definition of $d_M$. $\square$

Remark: When $D$ is the open unit disc centered in the origin and $d$ is the euclidean distance in $\mathbb{R}^2$ then the statement is: if $f$ is a continuous self-map of $D$ such that $f(D)$ is open in $D$ then there exists a rotation $r$ around the origin such that $r \circ f$ has a fixed point in $D$. If $f(D)$ is not open then the statement is false: take for example $f(x_1, x_2) = (1 + x_1^2 + x_2^2)/2$. 