Problem 10998
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Let D be a non-empty, open, connected, and relatively compact set in a metric space X with metric d. Prove that if f is a continuous map from D into D such that $f(D)$ is open, then there exists a point $x_0 \in D$ such that

$$d(x_0, \partial D) = d(f(x_0), \partial D).$$

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Assume that there is no such point x_0 then we will get a contradiction. The two sets

$$D^+ := \{x \in D\colon d(f(x), \partial D) > d(x, \partial D)\}, D^- := \{x \in D\colon d(f(x), \partial D) < d(x, \partial D)\}$$

are disjoint, open (f and $d(\cdot, \partial D)$ are continuous maps) and their union is equal to D. Since D is connected, one of these two sets is empty.
The set $\overline{D}$ is compact, hence there is a point $c \in \overline{D}$ (it is not unique in general) such that

$$d(c, \partial D) = d_M := \max_{x \in \overline{D}} d(x, \partial D).$$

Moreover, $d_M > 0$ because D is open and non-empty. Thus c stays inside D and since it reaches the maximal distance from the boundary, it belongs to D^-. Therefore the set D^+ has to be empty and $D = D^-$. This fact gives that for all $r > 0$, $f^{-1}(D_r) \subset D_r$, where D_r is the compact set $\{x \in D : d(x, \partial D) \geq r\}$.
We claim that the range set $f(D)$ is closed in D. Let $\{y_n\}_{n \geq 0}$ be a sequence in $f(D)$ which converges to $y \in D$ and let r be the distance of the compact set $\{y_n\}_{n \geq 0} \cup \{y\}$ from the boundary ∂D. Therefore $\{y_n\}_{n \geq 0} \cup \{y\} \subset D_r$ and the inclusion $f^{-1}(D_r) \subset D_r$ implies that if $\{x_n\}_{n \geq 0}$ is a sequence in D such that $f(x_n) = y_n$ then it is contained in D_r and, since D_r is compact, it converges, up to a subsequence, to some $x \in D$. Hence, by the continuity of f, $y = f(x) \in f(D)$.
So $f(D)$ is a non-empty open and closed subset of the connected set D that is $f(D) = D$. In particular there exists $x \in D$ such that $f(x) = c$ and

$$d_M = d(c, \partial D) = d(f(x), \partial D) < d(x, \partial D).$$

against the definition of d_M. □

Remark: When D is the open unit disc centered in the origin and d is the euclidean distance in $\mathbb{R}^2$ then the statement is: *if f is a continuous self-map of D such that $f(D)$ is open in D then there exists a rotation r around the origin such that $r \circ f$ has a fixed point in D.* If $f(D)$ is not open then the statement is false: take for example $f(x_1, x_2) = (1 + x_1^2 + x_2^2)/2$.