Problem 10991
(American Mathematical Monthly, Vol.110, February 2003)

Proposed by R. Mortini (France).

For complex $a, z \in \mathbb{D} = \{s : |s| < 1\}$, let $F(a,z) = (a+z)/(1+\overline{a}z)$ be a map of $\mathbb{D}$ onto $\mathbb{D}$. Let $\rho(a,b) = |(a-b)/(1-\overline{a}b)|$ be the pseudohyperbolic distance.
(**a**) *Prove that there exists a function $C : \mathbb{D} \to \mathbb{R}^+$ so that*

$$\rho(F(a,z), F(b,z)) \leq C(z)\rho(a,b)$$

for every $a, b, z \in \mathbb{D}$.
(**b**) *Find the minimal value of $C(z)$ for which this bound holds.*

Solution proposed by Roberto Tauraso, Dipartimento di Matematica, Università di Roma "Tor Vergata", via della Ricerca Scientifica, 00133 Roma, Italy.

It is well known that the map $F(a,z)$ is a ρ-isometry of the unit disc $\mathbb{D}$ and

$$\rho(z,a)^2 = |F(-a,z)|^2 = 1 - \frac{(1-|z|^2)(1-|a|^2)}{|1-\overline{a}z|^2}.$$

Moreover

$$\begin{aligned} F(-b, F(a,z)) &= \frac{F(a,z)-b}{1-\overline{b}F(a,z)} = \frac{z+a-b(1+\overline{a}z)}{(1+\overline{a}z)-\overline{b}(z+a)} \\ &= \frac{(1-\overline{a}b)z + (a-b)}{1-a\overline{b}+\overline{(a-b)}z} \\ &= e^{i\theta}F(c,z) \end{aligned}$$

where $c = (a-b)/(1-\overline{a}b)$ and $e^{i\theta} = (1-\overline{a}b)/(1-a\overline{b})$. In particular, when $a = b$, this implies that $F^{-1}(b,z) = F(-b,z)$.

We claim that it is possible to take $C(z) = (1+|z|)/(1-|z|)$ and this function is the best one. Indeed, we will show that for every $z \in \mathbb{D}$

$$C(z) := \sup\left\{\frac{\rho(F(a,z),F(b,z))}{\rho(a,b)} \,:\, a \neq b \in \mathbb{D}\right\} = \frac{1+|z|}{1-|z|}.$$

(**a**) We first prove that the inequality $\leq$ holds. Note that

$$\begin{aligned} \rho(F(a,z),F(b,z)) &= \rho(F(-b,F(a,z)), F(-b,F(b,z))) \\ &= \rho(e^{i\theta}F(c,z), z) = \rho(F(c,z), e^{-i\theta}z) \\ &\leq \rho(F(c,z),z) + \rho(z, e^{-i\theta}z). \end{aligned}$$

We consider the first term $\rho(F(c,z),z)$:

$$\begin{aligned} \rho(F(c,z),z)^2 &= 1 - \frac{(1-|z|^2)(1-|F(c,z)|^2)}{|1-\overline{z}F(c,z)|^2} = 1 - \frac{(1-|z|^2)^2(1-|c|^2)}{|1-\overline{z}F(c,z)|^2|1+\overline{c}z|^2} \\ &= 1 - \frac{(1-|z|^2)^2(1-|c|^2)}{|1+\overline{c}z-\overline{z}(z+c)|^2} = 1 - \frac{(1-|z|^2)^2(1-|c|^2)}{|1-|z|^2+2i\mathrm{Im}(\overline{c}z)|^2} \\ &= 1 - \frac{(1-|z|^2)^2(1-|c|^2)}{(1-|z|^2)^2+4|\mathrm{Im}(\overline{c}z)|^2} \leq 1 - \frac{(1-|z|^2)^2(1-|c|^2)}{(1-|z|^2)^2+4|c|^2|z|^2} \\ &\leq \frac{(1-|z|^2)^2+4|c|^2|z|^2-(1-|z|^2)^2(1-|c|^2)}{(1-|z|^2)^2+4|c|^2|z|^2} \\ &\leq \frac{(1+|z|^2)^2|c|^2}{(1-|z|^2)^2+4|c|^2|z|^2} \leq \left(\frac{1+|z|^2}{1-|z|^2}\right)^2 \cdot |c|^2. \end{aligned}$$

We consider the second term $\rho(z, e^{-i\theta}z)$:

$$\begin{aligned}\rho(z, e^{-i\theta}z) &= \left|\frac{z - e^{-i\theta}z}{1 - e^{i\theta}|z|^2}\right| \leq \frac{|z|}{1-|z|^2} \cdot \left|1 - e^{-i\theta}\right| \leq \frac{|z|}{1-|z|^2} \cdot \left|1 - \frac{1-a\overline{b}}{1-\overline{a}b}\right| \\ &\leq \frac{2|z|}{1-|z|^2} \cdot \frac{|\mathrm{Im}(a\overline{b})|}{|a-b|} \cdot |c| \leq \frac{2|z|}{1-|z|^2} \cdot |c|\end{aligned}$$

The last inequality holds because

$$|\mathrm{Im}(a\overline{b})| = |\mathrm{Im}(a\overline{b} - |b|^2)| = |\mathrm{Im}(\overline{b}(a-b))| \leq |\overline{b}(a-b)| \leq |\overline{b}| \cdot |a-b| \leq |a-b|.$$

Therefore, summing up the inequalities involving the two terms, we have that

$$\rho(F(a,z), F(b,z)) \leq \frac{1+|z|^2}{1-|z|^2} \cdot |c| + \frac{2|z|}{1-|z|^2} \cdot |c| \leq \frac{1+|z|}{1-|z|} \cdot |c|.$$

Since $|c| = \rho(a,b)$, we obtain

$$C(z) \leq \frac{1+|z|}{1-|z|}.$$

(**b**) We now prove that also the inequality $\geq$ holds. Since rotations around 0 are ρ-isometries then $C(z) = C(|z|)$ and therefore we can assume that $z \in [0,1)$. If $b = \overline{a}$ then

$$\begin{aligned}\rho(F(a,z), F(\overline{a},z))^2 &= 1 - \frac{(1-|F(a,z)|^2)(1-|F(\overline{a},z)|^2)}{|1-\overline{F(a,z)}F(\overline{a},z)|^2} \\ &= 1 - \frac{(1-|a|^2)^2(1-z^2)^2}{|(1+az)^2-(z+\overline{a})^2|^2}\end{aligned}$$

and

$$\rho(a,\overline{a})^2 = \left|\frac{a-\overline{a}}{1-a^2}\right|^2 = \frac{4|\mathrm{Im}(a)|^2}{|1-a^2|^2}.$$

Let $a = (1-s) + is^2$ with $s \in (0,1)$ then $a, b \in \mathbb{D}$ and they converge to 1 along two symmetric parabolic arcs as s goes to 0^+. Moreover

$$\begin{aligned}|1-a^2|^2 &= |2s - s^2 + s^4 - 2is^2(1-s)|^2 = 4s^2 - 4s^3 + 5s^4 + o(s^4), \\ (1-|a|^2)^2 &= (2s - s^2 - s^4)^2 = 4s^2 - 4s^3 + s^4 + o(s^4)\end{aligned}$$

and after some easy calculations

$$\begin{aligned}|(1+az)^2 - (z+\overline{a})^2|^2 &= |(1+az)+(z+\overline{a})|^2 \cdot |(1+az)-(z+\overline{a})|^2 \\ &= |(2-s)(1+z) - is^2(1-z)|^2 \cdot s^2|(1-z) + is(1+z)|^2 \\ &= (4s^2 - 4s^3 + s^4)(1-z^2)^2 + 4s^4(1+z)^4 + o(s^4).\end{aligned}$$

Therefore

$$\begin{aligned}\frac{\rho(F(a,z), F(\overline{a},z))^2}{\rho(a,\overline{a})^2} &= \left(1 - \frac{(1-|a|^2)^2(1-z^2)^2}{|(1+az)^2-(z+\overline{a})^2|^2}\right) \cdot \frac{|1-a^2|^2}{4|\mathrm{Im}(a)|^2} \\ &= \frac{4s^4(1+z)^4 + o(s^4)}{4s^2(1-z^2)^2 + o(s^2)} \cdot \frac{4s^2 + o(s^2)}{4s^4} = \frac{(1+z)^2}{(1-z)^2} + o(1).\end{aligned}$$

Hence for all $z \in \mathbb{D}$

$$C(z) = C(|z|) \geq \lim_{s \to 0^+} \frac{\rho(F(a,|z|), F(\overline{a},|z|))}{\rho(a,\overline{a})} = \frac{1+|z|}{1-|z|}.$$

□