Problem 10974
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The digital root $\rho(n)$ of a positive integer n is the eventual image of n under the mapping that carries an integer n to the sum of its base-ten digit. Thus $\rho(10974) = \rho(21) = 3$. Find $\rho(F_n)$, where F_n is the nth Fibonacci number, with $F_1 = F_2 = 1$.

Solution proposed by Roberto Tauraso, Dipartimento di Matematica, Università di Roma "Tor Vergata", via della Ricerca Scientifica, 00133 Roma, Italy.

It is well known that the digital root of a positive integer n is the remainder of the division of n by 9, so we define

$$f_n = \mathrm{mod}(F_n, 9) = \rho(F_n).$$

The sequence f_n can be generated by the recurrence relation

$$\begin{cases} f_{n+1} = f_n + f_{n-1} & \pmod 9 \\ f_1 = 1, \quad f_2 = 1 \end{cases}$$

and it is periodic with a period length of 24. Indeed

$$\begin{array}{llllll} f_1 = 1 & f_2 = 1 & f_3 = 2 & f_4 = 3 & f_5 = 5 & f_6 = 8 \\ f_7 = 4 & f_8 = 3 & f_9 = 7 & f_{10} = 1 & f_{11} = 8 & f_{12} = 0 \\ f_{13} = 8 & f_{14} = 8 & f_{15} = 7 & f_{16} = 6 & f_{17} = 4 & f_{18} = 1 \\ f_{19} = 5 & f_{20} = 6 & f_{21} = 2 & f_{22} = 8 & f_{23} = 1 & f_{24} = 0. \end{array}$$

and the sequence goes on with $f_{25} = f_1 = 1$ and $f_{26} = f_2 = 1$.
A first explicit formula for f_n is the following

$$f_n = \frac{(5+i)^n - (5-i)^n}{2i} = \mathrm{Im}\left((5+i)^n\right) \qquad \pmod 9.$$

Note that the complex numbers $5+i$ and $5-i$ allow to split in the Galois field GF(9) the characteristic polynomial $x^2 - x - 1$ associated to the Fibonacci recurrence equation. The formula can be proven by induction:

$$\begin{array}{ll} f_1 = \mathrm{Im}(5+i) = 1 & \pmod 9, \\ f_2 = \mathrm{Im}\left((5+i)^2\right) = \mathrm{Im}\left(6+i\right) = 1 & \pmod 9. \end{array}$$

moreover, for $n \geq 2$, we have that

$$\begin{aligned} f_n + f_{n-1} &= \mathrm{Im}\left((5+i)^n\right) + \mathrm{Im}\left((5+i)^{n-1}\right) && \pmod 9 \\ &= \mathrm{Im}\left((5+i)^n + (5+i)^{n-1}\right) && \pmod 9 \\ &= \mathrm{Im}\left((5+i)^{n-1} \cdot (5+i+1)\right) && \pmod 9 \\ &= \mathrm{Im}\left((5+i)^{n-1} \cdot (5+i)^2\right) = f_{n+1} && \pmod 9. \end{aligned}$$

Using the formula one can easily show that the sequence is periodic: since by the Fermat's Little Theorem $\mathrm{mod}(2^6, 9) = 1$, then

$$(5+i)^{24} = \left((5+i)^3\right)^8 = \left(2(1+i)\right)^8 = 2^8 \cdot 2^4 = 2^{12} = 1 \qquad \pmod 9$$

and for $n \geq 1$

$$f_{n+24} = \mathrm{Im}\left((5+i)^n \cdot (5+i)^{24}\right) = \mathrm{Im}\left((5+i)^n\right) = f_n \qquad \pmod 9.$$

Hence the computation of f_n can be further on simplified:

$$f_n = \mathrm{Im}\left((5+i)^{(n,24)}\right) \qquad \pmod 9$$

where $(n, 24) = \mathrm{mod}(n, 24)$ and, since

$$5 + i = \sqrt{26}\,\exp\left(i\,\arcsin\left(\tfrac{1}{\sqrt{26}}\right)\right),$$

we have our final formula

$$f_n = (\sqrt{26})^{(n,24)} \sin\left((n,24) \cdot \arcsin\left(\tfrac{1}{\sqrt{26}}\right)\right) \qquad \pmod 9.$$

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