Problem 10973
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Proposed by L. D. Servi (USA).

With $R_k(n)$ defined as below, prove that $\lim_{k\to\infty} R_k(2)/R_k(3) = 3/2$.

$$R_k(n) = \overbrace{\sqrt{2-\sqrt{2+\sqrt{2+\sqrt{2+\cdots+\sqrt{2+\sqrt{n}}}}}}}^{k \text{ square roots}}$$

Solution proposed by Roberto Tauraso, Dipartimento di Matematica, Università di Roma "Tor Vergata", via della Ricerca Scientifica, 00133 Roma, Italy.

Let $f(x) = \sqrt{2+x}$ then for any starting point $x \in (-2,2)$ the iterates $f^k(x)$ is an increasing sequence which converges to the attracting fixed point 2. Moreover

$$R_k(n) = \sqrt{2 - f^{k-1}(n-2)} \quad \text{and} \quad \lim_{k\to\infty} \frac{R_k(2)}{R_k(3)} = \lim_{k\to\infty} \sqrt{\frac{2-f^k(0)}{2-f^k(1)}}.$$

First note that

$$2 - f^{k-1}(x) = \frac{4-(f^{k-1}(x))^2}{2+f^{k-1}(x)} = \frac{4-(2+f^{k-2}(x))^2}{2+f^{k-1}(x)} = \frac{2-f^{k-2}(x)}{(f^k(x))^2}$$

and therefore going backward

$$2 - f^{k-1}(x) = \frac{2-f^{k-2}(x)}{(f^k(x))^2} = \frac{2-f^{k-3}(x)}{(f^k(x)f^{k-1}(x))^2} = \cdots = \frac{2-f^0(x)}{(f^k(x)f^{k-1}(x)\cdots f^2(x))^2}.$$

Since $2f'(x) = 1/\sqrt{2+x} = 1/f(x)$ then

$$2^k (f^k(x))' f(x) = \frac{1}{f^k(x)f^{k-1}(x)\cdots f^2(x)}$$

and we obtain the precious identity

$$4-(f^k(x))^2 = 2 - f^{k-1}(x) = \frac{2-x}{\left(f^k(x)f^{k-1}(x)\cdots f^2(x)\right)^2} = (4-x^2)\cdot(2^k(f^k(x))')^2.$$

Let $y_k(x) = f^k(x)$ and solve the following differential equation under the condition $y_k(2) = 2$

$$\sqrt{4-(y_k(x))^2} = \sqrt{4-x^2}\cdot 2^k\, y_k'(x).$$

Then

$$2^k \int_2^{y_k(x)} \frac{dy}{\sqrt{4-y^2}} = \int_2^x \frac{dx}{\sqrt{4-x^2}},$$

and integrating we find that

$$2^k \arccos\left(\frac{y_k(x)}{2}\right) = \arccos\left(\frac{x}{2}\right).$$

So we have an explicit formula for the k-th iterate of f

$$f^k(x) = y_k(x) = 2\cos\left(\frac{\arccos(\frac{x}{2})}{2^k}\right).$$

Now it is easy to compute our limit

$$\lim_{k\to\infty} \sqrt{\frac{2 - f^k(0)}{2 - f^k(1)}} = \lim_{k\to\infty} \sqrt{\frac{1 - \cos\left(\frac{\pi/2}{2^k}\right)}{1 - \cos\left(\frac{\pi/3}{2^k}\right)}} = \frac{3}{2}.$$

□

Remark: for $x \in [-2, 2]$ the map $f(x)$ has an inverse

$$f^{-1}(x) = 2\,T_2(\tfrac{x}{2}) \quad \text{for } x \in [0, 2]$$

where $T_2(x) = 2x^2 - 1$ is the Tchebycheff polynomial of degree 2.
It is well known that

$$T_2^k(x) = T_{2^k}(x) \quad \text{and} \quad T_n(x) = \cos(n \arccos(x)).$$

Hence

$$f^{-k}(x) = 2\,T_2^k(\tfrac{x}{2}) = 2\,T_{2^k}\left(\tfrac{x}{2}\right) = 2\cos\left(2^k \arccos\left(\tfrac{x}{2}\right)\right),$$

and therefore we find again the formula

$$f^k(x) = 2\cos\left(\frac{\arccos(\frac{x}{2})}{2^k}\right).$$