Problem 10958
(American Mathematical Monthly, Vol.109, August-September 2002)

Proposed by R. Chapman (UK).

Let A_n, be the n-by-n $0,1$-matrix with 1s in exactly those positions (j,k) such that $n \leq j+k \leq n+1$. Find the eigenvalues of A_n.

Solution proposed by Roberto Tauraso, Dipartimento di Matematica, Università di Roma "Tor Vergata", via della Ricerca Scientifica, 00133 Roma, Italy.

Let P_n be the characteristic polynomial of A_n

$$P_n(x) = \det(xI_n - A_n)$$

Expanding the determinant with respect to the last row of the matrix, it is simple to verify that the sequence of polynomials P_n satisfy the linear recurrence equation:

$$P_n(x) = x \cdot P_{n-1}(x) - P_{n-2}(x) \quad \text{with} \quad P_0(x) = 1,\ P_1(x) = x - 1.$$

By linearity, $T_n := P_n + P_{n-1}$ is another solution of the same recurrence equation but with different initial conditions:

$$T_n(x) = x \cdot T_{n-1}(x) - T_{n-2}(x) \quad \text{with} \quad T_0(x) = 2,\ T_1(x) = x.$$

Note that T_n is just the Tchebycheff polynomial of order n and each P_n can be written in the following form

$$P_n(x) = T_n(x) - P_{n-1}(x) = \cdots = (-1)^n \cdot \left[\sum_{k=1}^{n} (-1)^k T_k(x) + P_0(x)\right].$$

Since $T_n(2\cos\theta) = 2\cos(n\theta)$, then

$$P_n(2\cos\theta) = 2 \cdot (-1)^n \cdot \left[\sum_{k=1}^{n} (-1)^k \cos(k\theta) + \frac{1}{2}\right].$$

By the definition of the Dirichlet kernel $\delta_n(\theta)$

$$\delta_n(\theta) := \frac{\sin(n+\frac{1}{2})\theta}{2\pi \sin\frac{1}{2}\theta} = \frac{1}{\pi} \cdot \left[\sum_{k=1}^{n} \cos(k\theta) + \frac{1}{2}\right]$$

hence it follows that

$$\left[\sum_{k=1}^{n} (-1)^k \cos(k\theta) + \frac{1}{2}\right] = \pi \cdot \delta_n(\theta+\pi) = \frac{\sin(n+\frac{1}{2})(\theta+\pi)}{2\sin\frac{1}{2}(\theta+\pi)} = (-1)^n \cdot \frac{\cos(n+\frac{1}{2})\theta}{2\cos\frac{1}{2}\theta}.$$

Therefore

$$P_n(2\cos\theta) = \frac{\cos(n+\frac{1}{2})\theta}{\cos\frac{1}{2}\theta}.$$

Now it is easy to find the zeros of P_n (i. e. the eigenvalues of A_n):

$$x_k = 2\cos\theta_k = 2\cos\frac{2k+1}{2n+1}\pi \quad \text{for } k = 0 \ldots n-1.$$

□