Problem 10944

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Prove that if \( a, b, c \) are positive real numbers such that \( abc \geq 2^9 \), then

\[
\frac{1}{\sqrt{1 + a}} + \frac{1}{\sqrt{1 + b}} + \frac{1}{\sqrt{1 + c}} \geq \frac{3}{\sqrt[3]{1 + \sqrt[3]{abc}}}.
\]

Solution proposed by Roberto Tauraso, Dipartimento di Matematica, Università di Roma “Tor Vergata”, via della Ricerca Scientifica, 00133 Roma, Italy.

We will prove that the claim follows from these two inequalities

\[
\frac{1}{\sqrt{1 + x}} + \frac{1}{\sqrt{1 + y}} \geq \frac{2}{\sqrt{1 + \sqrt{xy}}} \quad \text{for } x, y > 0 \text{ and } xy \geq 9, \quad (1)
\]

\[
\frac{2}{\sqrt{1 + x}} + \frac{1}{\sqrt{1 + y}} \geq \frac{3}{\sqrt{1 + \sqrt{x^2y}}} \quad \text{for } x, y > 0 \text{ and } x^2y \geq 2^9. \quad (2)
\]

In fact, let \( a \geq b \geq c > 0 \) be such that \( abc \geq 2^9 \) then \( ab \geq 9 \) (otherwise \( abc < 27 \)) and we can apply (1) to the first two terms. Afterwards we use (2) with \( x = \sqrt{ab} \) and \( y = c \) completing the proof

\[
\frac{1}{\sqrt{1 + a}} + \frac{1}{\sqrt{1 + b}} + \frac{1}{\sqrt{1 + c}} \geq \frac{2}{\sqrt{1 + \sqrt{ab}}} + \frac{1}{\sqrt{1 + c}} \geq \frac{3}{\sqrt{1 + \sqrt[3]{abc}}}. 
\]

Now we prove that the inequalities (1) and (2) hold.

1) Let’s consider the function

\[
f(x) = \frac{1}{\sqrt{1 + x}} + \frac{1}{\sqrt{1 + p/x}} \quad \text{for } x > 0 \text{ and } p \geq 9
\]

then it suffices to show that

\[
\inf_{x>0} f(x) = f(\sqrt{p}) = \frac{2}{\sqrt{1 + \sqrt{p}}}
\]

The derivative of \( f \) is

\[
f'(x) = \frac{1}{2} \cdot (1 + x)^{-3/2} - \frac{1}{2} \cdot (1 + \frac{p}{x})^{-3/2} \cdot \left(-\frac{p}{x^2}\right)
\]

which is non-negative for \( x > 0 \) iff

\[
p^2(1 + x)^3 - x(x + p)^3 = (p - x^2)(x^2 - p(p - 3)x + p) \geq 0.
\]

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The condition $p \geq 9$ forces the second quadratic polynomials to have two distinct positive roots. Since their product is equal to $p$ then the smaller root belongs to the interval $(0, \sqrt{p})$ and the other one stays in $(\sqrt{p}, +\infty)$. So the sign of $f'$ is easily determined and it follows that, for $x > 0$, $f$ has a unique local minimum at $\sqrt{p}$. After checking the boundary behaviour of $f$, we can conclude that $\sqrt{p}$ is just the global minimum

$$
\lim_{x \to 0^+} f(x) = \lim_{x \to +\infty} f(x) = 1 \geq f(\sqrt{p}) = \frac{2}{\sqrt{1 + \sqrt{p}}} \quad \text{because } p \geq 9.
$$

2) Now we consider the function

$$
g(x) = \frac{2}{\sqrt{1 + x}} + \frac{1}{\sqrt{1 + q/x^2}} \quad \text{for } x > 0 \text{ and } q \geq 2^9
$$

then it suffices to show that

$$
\inf_{x > 0} g(x) = g(\sqrt[3]{q}) = \frac{3}{\sqrt{1 + \sqrt[3]{q}}}.
$$

Again we compute its derivative

$$
g'(x) = -(1 + x)^{-3/2} - (1 + \frac{q}{x^2})^{-3/2} \cdot \left(-\frac{q}{x^3}\right)
$$

which is non-negative for $x > 0$ iff

$$
q^{2/3}(1 + x) - (x^2 + q) = -x^2 + q^{2/3}x - (q - q^{2/3}) \geq 0.
$$

Since $q \geq 2^9$ then this quadratic polynomial has two distinct positive roots: the smaller one is $\sqrt[3]{q}$ and the other one is $\sqrt[3]{q}(\sqrt[3]{q} - 1)$. So the sign of $g'$ is easily determined and it follows that, for $x > 0$, $g$ has a unique local minimum at $\sqrt[3]{q}$. After checking the boundary behaviour of $g$, we can conclude that $\sqrt[3]{q}$ is just the global minimum

$$
\lim_{x \to 0^+} g(x) = 2 > \lim_{x \to +\infty} g(x) = 1 \geq g(\sqrt[3]{q}) = \frac{3}{\sqrt{1 + \sqrt[3]{q}}} \quad \text{because } q \geq 2^9.
$$