Problem 10944
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Prove that if a, b, c are positive real numbers such that $abc \geq 2^9$, then

$$\frac{1}{\sqrt{1+a}}+\frac{1}{\sqrt{1+b}}+\frac{1}{\sqrt{1+c}} \geq \frac{3}{\sqrt{1+\sqrt[3]{abc}}}$$

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We will prove that the claim follows from these two inequalities

$$\frac{1}{\sqrt{1+x}}+\frac{1}{\sqrt{1+y}} \geq \frac{2}{\sqrt{1+\sqrt{xy}}} \quad \text{for } x,y>0 \text{ and } xy \geq 9, \tag{1}$$

$$\frac{2}{\sqrt{1+x}}+\frac{1}{\sqrt{1+y}} \geq \frac{3}{\sqrt{1+\sqrt[3]{x^2y}}} \quad \text{for } x,y>0 \text{ and } x^2y \geq 2^9. \tag{2}$$

In fact, let $a \geq b \geq c > 0$ be such that $abc \geq 2^9$ then $ab \geq 9$ (otherwise $abc < 27$) and we can apply (1) to the first two terms. Afterwards we use (2) with $x = \sqrt{ab}$ and $y = c$ completing the proof

$$\frac{1}{\sqrt{1+a}}+\frac{1}{\sqrt{1+b}}+\frac{1}{\sqrt{1+c}} \geq \frac{2}{\sqrt{1+\sqrt{ab}}}+\frac{1}{\sqrt{1+c}} \geq \frac{3}{\sqrt{1+\sqrt[3]{abc}}}.$$

Now we prove that the inequalities (1) and (2) hold.

1) Let's consider the function

$$f(x)=\frac{1}{\sqrt{1+x}}+\frac{1}{\sqrt{1+p/x}} \quad \text{for } x>0 \text{ and } p \geq 9$$

then it suffices to show that

$$\inf_{x>0} f(x) = f(\sqrt{p}) = \frac{2}{\sqrt{1+\sqrt{p}}}.$$

The derivative of f is

$$f'(x) = -\frac{1}{2}\cdot(1+x)^{-3/2} - \frac{1}{2}\cdot(1+\frac{p}{x})^{-3/2}\cdot\left(-\frac{p}{x^2}\right)$$

which is non-negative for $x>0$ iff

$$p^2(1+x)^3 - x(x+p)^3 = (p-x^2)(x^2-p(p-3)x+p) \geq 0.$$

The condition $p \geq 9$ forces the second quadratic polynomials to have two distinct positive roots. Since their product is equal to p then the smaller root belongs to the interval $(0,\sqrt{p})$ and the other one stays in $(\sqrt{p},+\infty)$. So the sign of f' is easily determined and it follows that, for $x>0$, f has a unique local minimum at $\sqrt{p}$. After checking the boundary behaviour of f, we can conclude that $\sqrt{p}$ is just the global minimum

$$\lim_{x\to 0+} f(x) = \lim_{x\to+\infty} f(x) = 1 \geq f(\sqrt{p}) = \frac{2}{\sqrt{1+\sqrt{p}}} \quad \text{because } p \geq 9.$$

2) Now we consider the function

$$g(x) = \frac{2}{\sqrt{1+x}} + \frac{1}{\sqrt{1+q/x^2}} \quad \text{for } x > 0 \text{ and } q \geq 2^9$$

then it suffices to show that

$$\inf_{x>0} g(x) = g(\sqrt[3]{q}) = \frac{3}{\sqrt{1+\sqrt[3]{q}}}.$$

Again we compute its derivative

$$g'(x) = -(1+x)^{-3/2} - (1+\frac{q}{x^2})^{-3/2} \cdot \left(-\frac{q}{x^3}\right)$$

which is non-negative for $x > 0$ iff

$$q^{2/3}(1+x) - (x^2+q) = -x^2 + q^{2/3}x - (q - q^{2/3}) \geq 0.$$

Since $q \geq 2^9$ then this quadratic polynomial has two distinct positive roots: the smaller one is $\sqrt[3]{q}$ and the other one is $\sqrt[3]{q}(\sqrt[3]{q}-1)$. So the sign of g' is easily determined and it follows that, for $x > 0$, g has a unique local minimum at $\sqrt[3]{q}$. After checking the boundary behaviour of g, we can conclude that $\sqrt[3]{q}$ is just the global minimum

$$\lim_{x\to 0+} g(x) = 2 > \lim_{x\to +\infty} g(x) = 1 \geq g(\sqrt[3]{q}) = \frac{3}{\sqrt{1+\sqrt[3]{q}}} \quad \text{because } q \geq 2^9.$$

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