Problem 10930
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Proposed by P. Schweitzer (USA).

Show that there do not exist real 2-by-2 matices A and B such that their commutator is nonzero and commutes with both A and B. (The commutator C of A and B is defined by $C = AB - BA$.)

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By the Cayley-Hamilton theorem, the matrices A and B solve their characteristic polynomial:

$$A^2 = aA + b\,,\ \ B^2 = cB + d$$

where $a = \mathrm{tr}(A)$, $b = -\det(A)$ and $c = \mathrm{tr}(B)$, $d = -\det(B)$. Since C commutes with A

$$\begin{cases} AC - CA = 0 \\ AC + CA = A^2B + BA^2 = (aA + b)B - B(aA + b) = aC \end{cases}$$

Therefore, $AC = CA = \frac{a}{2}C$ and, in the same way, $BC = CB = \frac{c}{2}C$. Thus

$$C^2 = (AB - BA)C = A(BC) - B(AC) = \frac{a}{2} \cdot \frac{c}{2}C - \frac{c}{2} \cdot \frac{a}{2}C = 0.$$

Assume that $C \neq 0$. Then there is a vector $v \neq 0$ such that $u := Cv \neq 0$. Note that u is a common eigenvector of C, A and B:

$$Cu = C^2v = 0,\ Au = ACv = \frac{a}{2}Cv = \frac{a}{2}u,\ Bu = BCv = \frac{c}{2}Cv = \frac{c}{2}u.$$

The vectors u and v are linearly independent: let $\alpha, \beta \in \mathbb{R}$ such that $\alpha u + \beta v = 0$ then

$$0 = C(\alpha u + \beta v) = \alpha Cu + \beta Cv = \alpha C^2 v + \beta Cv = \beta u,$$

which implies that $\alpha = \beta = 0$. Since $a = \mathrm{tr}(A)$ and $c = \mathrm{tr}(B)$ then the representations of the matrices A and B with respect to the base $< u, v >$ are

$$A = \begin{pmatrix} \frac{a}{2} & \gamma \\ 0 & \frac{a}{2} \end{pmatrix} \text{ and } B = \begin{pmatrix} \frac{c}{2} & \delta \\ 0 & \frac{c}{2} \end{pmatrix}$$

for some $\gamma, \delta \in \mathbb{R}$. Then

$$C = AB - BA = \begin{pmatrix} \frac{ac}{4} & \frac{a}{2}\delta + \frac{c}{2}\gamma \\ 0 & \frac{ac}{4} \end{pmatrix} - \begin{pmatrix} \frac{ac}{4} & \frac{c}{2}\gamma + \frac{a}{2}\delta \\ 0 & \frac{ac}{4} \end{pmatrix} = 0$$

that is a contradiction. □

Remark. The statement does not hold in higher dimension: for example in $R^{3\times 3}$

$$\text{if } A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \text{ and } B = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \text{ then } C = AB - BA = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

and $AC = CA$, $BC = CB$.