Problem 10924
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A regular polygon of 2001 *sides is inscribed in a circle of unit radius. Prove that its side and all its diagonals have irrational lengths.*

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We will prove a more general statement:
In a regular polygon of $N \geq 2$ sides and inscribed in a circle of unit radius, the side and all the diagonals have irrational lengths iff N is an odd integer.

First recall that for all integers $n \geq 1$ and for all $\theta \in \mathbb{R}$

$$\cos n\theta + i \sin n\theta = (\cos\theta + i\sin\theta)^n = \sum_{k=0}^{n} \binom{n}{k} i^k (\sin\theta)^k (\cos\theta)^{n-k}.$$

After taking the imaginery part, we have

$$\sin n\theta = \sum_{k=1,\, k\,\mathrm{odd}}^{n} \binom{n}{k} (-1)^{\frac{k-1}{2}} (\sin\theta)^k (\cos\theta)^{n-k}.$$

If n is odd then the exponents $n-k$ are even and

$$(\cos\theta)^{n-k} = (\cos^2\theta)^{\frac{n-k}{2}} = (1-\sin^2\theta)^{\frac{n-k}{2}}.$$

Hence, for all odd integers $n \geq 1$, there is a polynomial $Q_n(x) \in \mathbb{Z}[x]$ of degree $n-1$, of the following form

$$Q_n(x) = (-1)^{\frac{n-1}{2}} \left(1 + \binom{n}{2}\right) x^{n-1} + \ldots + n,$$

such that

$$\sin n\theta = \sin\theta \cdot Q_n(\sin\theta) \quad \text{for all odd integers } n \geq 1. \tag{1}$$

After this key remark, we start the proof of our statement.
If N is even then the diagonal between two opposite vertices has length $2 \in \mathbb{Q}$. On the other hand, if N is odd then the side and the diagonals have lengths: $2\sin(\pi m/N)$ for $m = 1, \ldots, N-1$. Assume that one of these numbers is rational, then we will reach a contradiction.
Since $1 \leq m \leq N-1$, there are an odd prime p and some integer $e \geq 1$ such that p^e divides N, p^{e-1} divides m, but p^e does not divide m. Therefore N/p^e is an odd integer and by (1)

$$\sin\left(\frac{\pi\, m'}{p}\right) = \sin\left(\frac{N}{p^e} \cdot \frac{\pi\, m}{N}\right) = \sin\left(\frac{\pi\, m}{N}\right) \cdot Q_{\frac{N}{p^e}}\left(\sin\left(\frac{\pi\, m}{N}\right)\right) \in \mathbb{Q}$$

where $m' = m/p^{e-1}$. Since $\mathrm{MCD}(m', p) = 1$, if $k \in \mathbb{Z}$ then there are two integers s_0 and t_0 such that $s = s_0 + j\,p$ and $t = t_0 - j\,m'$ solve the following linear diophantine equation for all $j \in \mathbb{Z}$:

$$s\,m' + t\,p = k.$$

We can always pick j such that s is an odd positive integer. Then, by (1)

$$\sin\left(\frac{\pi\, k}{p}\right) = \sin\left(\frac{\pi(s\,m' + t\,p)}{p}\right) = (-1)^t \sin\left(\frac{\pi\, m'}{p}\right) \cdot Q_s\left(\sin\left(\frac{\pi\, m'}{p}\right)\right) \in \mathbb{Q}.$$

Varying $k = \pm 1, \ldots, \pm \frac{p-1}{2}$, we obtain $p-1$ distinct rational numbers which are the zeroes of the polynomial

$$Q_p(x) = (-1)^{\frac{p-1}{2}} \left(1 + \binom{p}{2}\right) x^{p-1} + \ldots + p.$$

The polynomial Q_p has integer coefficients and therefore every rational zero has the following property: the numerator divides p whereas the denominator divides $a = 1 + p(p-1)/2$. It follows that

$$S = \left\{ \sin\left(\frac{\pi\, k}{p}\right) \;:\; k = \pm 1, \ldots, \pm \frac{p-1}{2} \right\} \subset T = \left\{ \pm \frac{1}{d}, \pm \frac{p}{d} \;:\; d \mid a \right\} \cap (-1, 1).$$

Hence

$$\cos\left(\frac{\pi}{2p}\right) = \sin\left(\frac{\pi}{p} \cdot \frac{(p-1)}{2}\right) = \max S \le \max T \le \frac{p}{p+1} = 1 - \frac{1}{p+1},$$

and therefore

$$\frac{1}{p+1} \le 1 - \cos\left(\frac{\pi}{2p}\right) = 2 \sin^2\left(\frac{\pi}{4p}\right) < 2 \left(\frac{\pi}{4p}\right)^2 < \frac{2}{p^2}.$$

This inequality never holds for $p \ge 3$ which is a contradiction. $\square$

Note that, if $N = 2001 = 3 \cdot 23 \cdot 29$ then we can get a contradiction in a different way:

if $p = 23$ then $a = 254 = 2 \cdot 127$ and $T = \{\pm\frac{1}{2},\ \pm\frac{1}{127},\ \pm\frac{1}{254}\ \pm\frac{23}{127},\ \pm\frac{23}{254}\}$,
if $p = 29$ then $a = 407 = 11 \cdot 37$ and $T = \{\pm\frac{1}{11},\ \pm\frac{1}{37},\ \pm\frac{1}{407},\ \pm\frac{29}{37}, \pm\frac{29}{407}\}$.

In both cases the number of elements of T is much less than $p-1$. The remaining case $p = 3$ is easily solved because $\sin(\pi/3) = \sqrt{3}/2 \notin \mathbb{Q}$. $\square$