Problem 10824
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Suppose P is a point in the interior of triangle ABC such that $\angle PAB = \angle PBC = \angle PCA = 30°$. Prove that ABC is equilateral.

Solution proposed by Roberto Tauraso, Dipartimento di Matematica, Università di Roma "Tor Vergata", via della Ricerca Scientifica, 00133 Roma, Italy.

Let $X = \angle PAC$, $Y = \angle PBA$ and $Z = \angle PCB$. We are going to show that $X = Y = Z = 30°$. By the Law of Sines

$$\frac{|PC|}{\sin X} = \frac{|PA|}{\sin 30°}, \quad \frac{|PA|}{\sin Y} = \frac{|PB|}{\sin 30°}, \quad \frac{|PB|}{\sin Z} = \frac{|PC|}{\sin 30°},$$

and therefore it easily follows that

$$(\sin X \cdot \sin Y \cdot \sin Z)^{1/3} = \sin 30°.$$

Moreover, by the arithmetic-mean-geometric-mean inequality and since the function $\sin x$ is concave in $(0°, 90°)$

$$\sin 30° = (\sin X \cdot \sin Y \cdot \sin Z)^{1/3} \leq \frac{\sin X + \sin Y + \sin Z}{3} \leq \sin\left(\frac{X + Y + Z}{3}\right).$$

But also the last term is equal to $\sin 30°$ because $X + Y + Z = 90°$. Hence the geometric mean and the arithmetic mean are equal and this happens if only if all the involved elements are equal. Therefore $\sin X = \sin Y = \sin Z$ that is $X = Y = Z = 30°$. $\square$