Problem 10824

Proposed by Ho-joo Lee (South Korea).

Suppose \( P \) is a point in the interior of triangle \( ABC \) such that \( \angle PAB = \angle PBC = \angle PCA = 30^\circ \). Prove that \( ABC \) is equilateral.

Solution proposed by Roberto Tauraso, Dipartimento di Matematica, Università di Roma “Tor Vergata”, via della Ricerca Scientifica, 00133 Roma, Italy.

Let \( X = \angle PAC \), \( Y = \angle PBA \) and \( Z = \angle PCB \). We are going to show that \( X = Y = Z = 30^\circ \). By the Law of Sines

\[
\frac{|PC|}{\sin X} = \frac{|PA|}{\sin 30^\circ}, \quad \frac{|PA|}{\sin Y} = \frac{|PB|}{\sin 30^\circ}, \quad \frac{|PB|}{\sin Z} = \frac{|PC|}{\sin 30^\circ},
\]

and therefore it easily follows that

\[
(sin X \cdot sin Y \cdot sin Z)^{1/3} = \sin 30^\circ.
\]

Moreover, by the arithmetic-mean-geometric-mean-inequality and since the function \( \sin x \) is concave in \((0^\circ, 90^\circ)\)

\[
\sin 30^\circ = (sin X \cdot sin Y \cdot sin Z)^{1/3} \leq \frac{\sin X + \sin Y + \sin Z}{3} \leq \sin \left( \frac{X + Y + Z}{3} \right).
\]

But also the last term is equal to \( \sin 30^\circ \) because \( X + Y + Z = 90^\circ \). Hence the geometric mean and the arithmetic mean are equal and this happens if only if all the involved elements are equal. Therefore \( sin X = sin Y = sin Z \) that is \( X = Y = Z = 30^\circ \). \( \square \)