Problem 10739
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Proposed by O. Ciaurri (Spain).

Suppose that $f : [0,1] \to \mathbb{R}$ *has a continuous second derivative with* $f''(x) > 0$ *on* $(0,1)$*, and suppose that* $f(0) = 0$*. Choose* $a \in (0,1)$ *such that* $f'(a) < f(1)$*. Show that there is a unique* $b \in (a,1)$ *such that* $f'(a) = f(b)/b$*.*

Solution proposed by Roberto Tauraso, Dipartimento di Matematica, Università di Firenze, viale Morgagni 67/A, 50134 Firenze, Italy.

It is sufficient to assume that f is continuous on $[0,1]$ and it has a strictly increasing first derivative $f'(x)$ on $(0,1)$. We define the map $R(x) = f(x)/x$ for $x \in (0,1]$. Let $x \in (0,1)$ then, by the Mean Value Theorem applied to the map f with respect to the interval $[0,x]$, there exists $x_0 \in (0,x)$ such that

$$R(x) = \frac{f(x) - f(0)}{x - 0} = f'(x_0) < f'(x)$$

where the last inequality follows from the fact that f' is strictly increasing on $(0,1)$. Now let $a \in (0,1)$ be such that $f'(a) < f(1)$ then

$$R(a) < f'(a) < R(1) = f(1)$$

which implies that there is a unique $b \in (a,1)$ such that $R(b) = f(b)/b = f'(a)$ because R is continuous and strictly increasing on $(0,1]$:

$$R'(x) = \frac{xf'(x) - f(x)}{x^2} = \frac{f'(x) - R(x)}{x} > 0 \quad \text{for all } x \in (0,1).$$

□