Problem 10711
(American Mathematical Monthly, Vol.106, February 1999)

Proposed by F. Luca (Germany).

A natural number is perfect *if it is the sum of its proper divisors. Prove that two consecutive numbers cannot both be perfect.*

Solution proposed by Roberto Tauraso, Dipartimento di Matematica, Università di Firenze, viale Morgagni 67/A, 50134 Firenze, Italy.

We denote by $\sigma(n)$ the sum of the divisors of the natural number n. It is easy to see that the arithmetic function σ is multiplicative that is

$$\sigma(a \cdot b) = \sigma(a) \cdot \sigma(b) \quad \text{iff } (a, b) = 1.$$

Then n is a perfect number if and only if $\sigma(n) = 2n$. Moreover the following two elementary properties of perfect numbers hold:

(i) if n is even then n is perfect if and only if

$$n = 2^{q-1}(2^q - 1)$$

where q and $2^q - 1$ are primes.

(ii) if n is an odd perfect number then it must have the form

$$n = p^{4m+1} s^2$$

where p is a prime such that $[p]_4 = 1$ and $(p, s) = 1$.

For a proof of the above results see for example L. Dickson, *History of the theory of numbers*, vol. 1, 1919, pp. 1-33. Note that the question as to whether there exist odd perfect numbers is as yet unsolved.

Therefore, by (i), two consecutive numbers are both perfect if

$$n_- = 2^{q-1}(2^q - 1) - 1 \quad \text{or} \quad n_+ = 2^{q-1}(2^q - 1) + 1$$

is perfect. The prime q is not 2 because neither $n_- = 5$ nor $n_+ = 7$ is perfect. Hence $q = 2a + 1 \geq 3$.

1) Assume that n_- is perfect. Then

$$[n_-]_4 = [4^a]_4 \cdot [2 \cdot 4^a - 1]_4 + [-1]_4 = [3]_4.$$

By (ii), $n_- = p^{4m+1} s^2$ with $[p]_4 = 1$. Since s is odd then either $[s]_4 = 1$ or $[s]_4 = 3$ and in both cases $[s^2]_4 = 1$. So we have the contradiction

$$[3]_4 = [n_-]_4 = [p]_4^{4m+1} \cdot [s^2]_4 = [1]_4.$$

2) Assume that n_+ is perfect. Then

$$[n_+]_3 = [4^a]_3 \cdot [2 \cdot 4^a - 1]_3 + [1]_3 = [1]_3 + [1]_3 = [2]_3.$$

By (ii), $n_+ = p^{4m+1} s^2$ where p is a prime such that $(p, s) = 1$. Now, it easy to verify that

$$[p^{4m+1}]_3 \cdot [s^2]_3 = [2]_3 \quad \text{iff } [p]_3 = [2]_3 \text{ and } [s^2]_3 = 1.$$

Moreover, since p is prime,

$$[\sigma(p^{4m+1})]_3 = [1 + p + \ldots + p^{4m} + p^{4m+1}]_3 = [1 + 2 + \ldots + 1 + 2]_3 = [0]_3.$$

By the multiplicative property of σ, we have the contradiction

$$[1]_3 = [2n_+]_3 = [\sigma(n_+)]_3 = [\sigma(p^{4m+1})]_3 \cdot [\sigma(s^2)]_3 = [0]_3.$$

□