Problem 10704
(American Mathematical Monthly, Vol.106, January 1999)

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Show that there are infinitely many pairs $((a,b,c),(a',b',c'))$ *of primitive Pythagorean triples such that* $|a-a'|$, $|b-b'|$, *and* $|c-c'|$ *are all equal to* 3 *or* 4. *Examples include* $((12,5,13),(15,8,17))$ *and* $((77,36,85),(80,39,89))$.

Solution proposed by Roberto Tauraso, Dipartimento di Matematica, Università di Firenze, viale Morgagni 67/A, 50134 Firenze, Italy.

The complete set of primitive Pythagorean triples can be generated in the following way:

$$(a,b,c)=(rs,\frac{1}{2}(r^2-s^2),\frac{1}{2}(r^2+s^2)) \qquad (1)$$

where (r,s) is a pair of odd integers such that $\gcd(r,s)=1$ and $r>s\geq 1$. Let (a,b,c) be a primitive Pythagorean triple, then

$$(a+3)^2+(b+3)^2=(c+4)^2 \qquad (2)$$

if and only if $4c-3a-3b=1$, that is, by (**??**),

$$r^2-6rs+7s^2=2. \qquad (3)$$

Given an odd integer $s\geq 1$ and solving with respect to r, we find that r is an odd integer too if there exist two integers $k\geq 0$ and $q\geq 1$ such that

$$s=2k+1 \quad \text{and} \quad r=3s\pm\sqrt{2(s^2+1)}=3s\pm 2q. \qquad (4)$$

that is

$$k^2+(k+1)^2=q^2. \qquad (5)$$

Now we consider the sequence of pairs $\{(k_n,q_n)\}_{n\geq 0}$ defined by induction as follows

$$\begin{pmatrix}k_0\\q_0\end{pmatrix}=\begin{pmatrix}0\\1\end{pmatrix} \quad \text{and} \quad \begin{pmatrix}k_{n+1}\\q_{n+1}\end{pmatrix}=\begin{bmatrix}3&2\\4&3\end{bmatrix}\begin{pmatrix}k_n\\q_n\end{pmatrix}+\begin{pmatrix}1\\2\end{pmatrix} \quad \text{for } n\geq 0.$$

It is easy to see that the elements of this sequence verify equation (**??**):

$$k_n^2+(k_n+1)^2=q_n^2 \quad \text{for } n\geq 0.$$

Therefore, following (**??**), we set

$$\begin{pmatrix}r_n\\s_n\end{pmatrix}=2\begin{bmatrix}3&\pm 1\\1&0\end{bmatrix}\begin{pmatrix}k_n\\q_n\end{pmatrix}+\begin{pmatrix}3\\1\end{pmatrix} \quad \text{for } n\geq 0.$$

Since by the above matrix equation

$$\begin{pmatrix}k_n\\q_n\end{pmatrix}=\frac{1}{2}\begin{bmatrix}0&1\\\pm 1&\mp 3\end{bmatrix}\begin{pmatrix}r_n\\s_n\end{pmatrix}-\frac{1}{2}\begin{pmatrix}1\\0\end{pmatrix} \quad \text{for } n\geq 0$$

then, after some calculations, we can write down two sequences of pairs $\{(r_n^+,s_n^+)\}_{n\geq 0}$ and $\{(r_n^-,s_n^-)\}_{n\geq 1}$ in an inductive form:

$$\begin{pmatrix}r_0^+\\s_0^+\end{pmatrix}=\begin{pmatrix}5\\1\end{pmatrix} \quad \text{and} \quad \begin{pmatrix}r_{n+1}^+\\s_{n+1}^+\end{pmatrix}=\begin{bmatrix}9&-14\\2&-3\end{bmatrix}\begin{pmatrix}r_n^+\\s_n^+\end{pmatrix} \quad \text{for } n\geq 0,$$

$$\begin{pmatrix}r_1^-\\s_1^-\end{pmatrix}=\begin{pmatrix}11\\7\end{pmatrix} \quad \text{and} \quad \begin{pmatrix}r_{n+1}^-\\s_{n+1}^-\end{pmatrix}=\begin{bmatrix}-3&14\\-2&9\end{bmatrix}\begin{pmatrix}r_n^-\\s_n^-\end{pmatrix} \quad \text{for } n\geq 1.$$

The elements of these two sequences generate, by (**??**), infinite Pythagorean triples (a, b, c) which verify (**??**) and therefore solve our problem. Note that the triples generated by the first elements $(5, 1)$ and $(11, 7)$ are respectively just $(5, 12, 13)$ and $(77, 36, 85)$.

Now we prove that actually all these Pythagorean triples are primitive. We consider each sequence separately:

(i) The integers r_n^+, s_n^+ are all odd and $\gcd(r_n^+, s_n^+) = 1$ because

$$\gcd(5, 1) = 1 \quad \text{and} \quad \det(\begin{bmatrix} 9 & -14 \\ 2 & -3 \end{bmatrix}) = 1.$$

Moreover, $s_{n+1}^+ > s_n^+$ and $r_n^+ > 2s_n^+$. Indeed these are true for $n = 0$ and by induction, for $n \geq 0$,

$$s_{n+1}^+ = 2r_n^+ - 3s_n^+ > s_n^+,$$
$$r_{n+1}^+ - 2s_{n+1}^+ = 5r_n^+ - 8s_n^+ > 2s_n^+ > 0.$$

Therefore $r_n^+ > s_n^+ \geq 1$ for all $n \geq 0$ as requested.

(ii) The integers r_n^-, s_n^- are all odd and $\gcd(r_n^-, s_n^-) = 1$ because

$$\gcd(11, 7) = 1 \quad \text{and} \quad \det(\begin{bmatrix} -3 & 14 \\ -2 & 9 \end{bmatrix}) = 1.$$

Moreover, $s_{n+1}^- > s_n^-$ and $r_n^- < 2s_n^-$. Indeed these are true for $n = 1$ and by induction, for $n \geq 1$,

$$s_{n+1}^- = -2r_n^- + 9s_n^- > 5s_n^- > s_n^-,$$
$$2s_{n+1}^- - r_{n+1}^- = 4s_n^- - r_n^- > 2s_n^- > 0.$$

Therefore $r_n^- > s_n^- \geq 1$ for all $n \geq 1$ as requested because

$$11 > 7 \text{ and } r_{n+1}^- - s_{n+1}^- = 5s_n^- - r_n^- > 3s_n^- > 0.$$

$\square$