Problem 10704

Proposed by W. G. Spohn (USA).

Show that there are infinitely many pairs \((a, b, c), (a', b', c')\) of primitive Pythagorean triples such that \(|a - a'|, |b - b'|, \text{ and } |c - c'|\) are all equal to 3 or 4. Examples include \(((12, 5, 13), (15, 8, 17))\) and \(((77, 36, 85), (80, 39, 89))\).

Solution proposed by Roberto Tauraso, Dipartimento di Matematica, Università di Firenze, viale Morgagni 67/A, 50134 Firenze, Italy.

The complete set of primitive Pythagorean triples can be generated in the following way:

\[(a, b, c) = (rs, \frac{1}{2}(r^2 - s^2), \frac{1}{2}(r^2 + s^2))\]  \hspace{1cm} (1)

where \((r, s)\) is a pair of odd integers such that \(\gcd(r, s) = 1\) and \(r > s \geq 1\). Let \((a, b, c)\) be a primitive Pythagorean triple, then

\[(a + 3)^2 + (b + 3)^2 = (c + 4)^2\]  \hspace{1cm} (2)

if and only if \(4c - 3a - 3b = 1\), that is, by (??),

\[r^2 - 6rs + 7s^2 = 2.\]  \hspace{1cm} (3)

Given an odd integer \(s \geq 1\) and solving with respect to \(r\), we find that \(r\) is an odd integer too if there exist two integers \(k \geq 0\) and \(q \geq 1\) such that

\[s = 2k + 1 \quad \text{and} \quad r = 3s + \sqrt{2(s^2 + 1)} = 3s \pm 2q.\]  \hspace{1cm} (4)

that is

\[k^2 + (k + 1)^2 = q^2.\]  \hspace{1cm} (5)

Now we consider the sequence of pairs \(\{(k_n, q_n)\}_{n \geq 0}\) defined by induction as follows

\[
\begin{pmatrix}
k_0 \\
q_0
\end{pmatrix}
= \begin{pmatrix}
0 \\
1
\end{pmatrix}
\quad \text{and} \quad
\begin{pmatrix}
k_{n+1} \\
q_{n+1}
\end{pmatrix}
= \begin{pmatrix}
3 & 2 \\
4 & 3
\end{pmatrix}
\begin{pmatrix}
k_n \\
q_n
\end{pmatrix}
+ \begin{pmatrix}
1 \\
2
\end{pmatrix}
\quad \text{for } n \geq 0.
\]

It is easy to see that the elements of this sequence verify equation (??):

\[k_n^2 + (k_n + 1)^2 = q_n^2 \quad \text{for } n \geq 0.
\]

Therefore, following (??), we set

\[
\begin{pmatrix}
r_n \\
s_n
\end{pmatrix}
= \begin{pmatrix}
3 & \pm 1 \\
1 & 0
\end{pmatrix}
\begin{pmatrix}
k_n \\
q_n
\end{pmatrix}
+ \begin{pmatrix}
3 \\
1
\end{pmatrix}
\quad \text{for } n \geq 0.
\]

Since by the above matrix equation

\[
\begin{pmatrix}
k_n \\
q_n
\end{pmatrix}
= \frac{1}{2}
\begin{pmatrix}
0 & 1 \\
\pm 3 & 0
\end{pmatrix}
\begin{pmatrix}
r_n \\
s_n
\end{pmatrix}
- \frac{1}{2}
\begin{pmatrix}
1 \\
0
\end{pmatrix}
\quad \text{for } n \geq 0,
\]

then, after some calculations, we can write down two sequences of pairs \(\{(r_n^+, s_n^+)\}_{n \geq 0}\) and \(\{(r_n^-, s_n^-)\}_{n \geq 1}\)

in an inductive form:

\[
\begin{pmatrix}
r_0^+ \\
s_0^+
\end{pmatrix}
= \begin{pmatrix}
5 \\
1
\end{pmatrix}
\quad \text{and} \quad
\begin{pmatrix}
r_{n+1}^+ \\
s_{n+1}^+
\end{pmatrix}
= \begin{pmatrix}
9 & -14 \\
2 & -3
\end{pmatrix}
\begin{pmatrix}
r_n^+ \\
s_n^+
\end{pmatrix}
\quad \text{for } n \geq 0,
\]

\[
\begin{pmatrix}
r_1^- \\
s_1^-
\end{pmatrix}
= \begin{pmatrix}
11 \\
7
\end{pmatrix}
\quad \text{and} \quad
\begin{pmatrix}
r_{n+1}^- \\
s_{n+1}^-
\end{pmatrix}
= \begin{pmatrix}
-3 & 14 \\
-2 & 9
\end{pmatrix}
\begin{pmatrix}
r_n^- \\
s_n^-
\end{pmatrix}
\quad \text{for } n \geq 1.
The elements of these two sequences generate, by (??), infinite Pythagorean triples \((a, b, c)\) which verify (??) and therefore solve our problem. Note that the triples generated by the first elements \((5, 1)\) and \((11, 7)\) are respectively just \((5, 12, 13)\) and \((77, 36, 85)\).

Now we prove that actually all these Pythagorean triples are primitive. We consider each sequence separately:

(i) The integers \(r_n^+, s_n^+\) are all odd and \(\gcd(r_n^+, s_n^+) = 1\) because

\[
\gcd(5, 1) = 1 \quad \text{and} \quad \det(\begin{bmatrix} 9 & -14 \\ 2 & -3 \end{bmatrix}) = 1.
\]

Moreover, \(s_{n+1}^+ > s_n^+\) and \(r_{n+1}^+ > 2s_n^+.\) Indeed these are true for \(n = 0\) and by induction, for \(n \geq 0,\)

\[
s_{n+1}^+ = 2r_n^+ - 3s_n^+ > s_n^+,
\]

\[
r_{n+1}^+ - 2s_{n+1}^+ = 5r_n^+ - 8s_n^+ > 2s_n^+ > 0.
\]

Therefore \(r_n^+ > s_n^+ \geq 1\) for all \(n \geq 0\) as requested.

(ii) The integers \(r_n^-, s_n^-\) are all odd and \(\gcd(r_n^-, s_n^-) = 1\) because

\[
\gcd(11, 7) = 1 \quad \text{and} \quad \det(\begin{bmatrix} -3 & 14 \\ -2 & 9 \end{bmatrix}) = 1.
\]

Moreover, \(s_{n+1}^- > s_n^-\) and \(r_{n+1}^- < 2s_n^-\) Indeed these are true for \(n = 1\) and by induction, for \(n \geq 1,\)

\[
s_{n+1}^- = -2r_n^- + 9s_n^- > 5s_n^- > s_n^-,
\]

\[
2s_{n+1}^- - r_{n+1}^- = 4s_n^- - r_n^- > 2s_n^- > 0.
\]

Therefore \(r_n^- > s_n^- \geq 1\) for all \(n \geq 1\) as requested because

\[
11 > 7 \quad \text{and} \quad r_{n+1}^- - s_{n+1}^- = 5s_n^- - r_n^- > 3s_n^- > 0.
\]

\(\square\)