Problem 10697
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Proposed by J. L. Diaz-Barrero (Spain).

Given n distinct nonzero complex numbers $z_1, z_2, \ldots, z_n$, show that

$$\sum_{k=1}^{n} \frac{1}{z_k} \prod_{\substack{j=1 \\ j \neq k}}^{n} \frac{1}{z_k - z_j} = \frac{(-1)^{n+1}}{z_1 z_2 \cdots z_n}.$$

Solution proposed by Roberto Tauraso, Dipartimento di Matematica, Università di Firenze, viale Morgagni 67/A, 50134 Firenze, Italy.

The polynomial

$$P(z) = \sum_{k=1}^{n} \prod_{\substack{j=1 \\ j \neq k}}^{n} \frac{z - z_j}{z_k - z_j} - 1,$$

whose degree is less than n, has at least n distinct roots, i.e. $P(z_i) = 0$ for $i = 1, 2 \ldots, n$. Therefore, by the Fundamental Theorem of Algebra, the polynomial P is identically zero. Thus

$$1 = 1 + P(0) = \sum_{k=1}^{n} \prod_{\substack{j=1 \\ j \neq k}}^{n} \frac{-z_j}{z_k - z_j} = (-1)^{n-1} z_1 z_2 \cdots z_n \cdot \sum_{k=1}^{n} \frac{1}{z_k} \prod_{\substack{j=1 \\ j \neq k}}^{n} \frac{1}{z_k - z_j}$$

and we easily find our identity after dividing both sides of the equality by the nonzero complex number $(-1)^{n-1} z_1 z_2 \cdots z_n$. □